C3M16

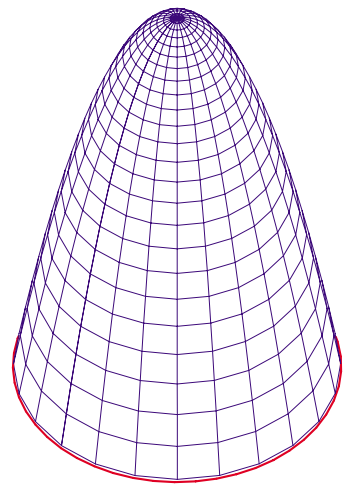
Stoke's Theorem

STOKE'S THEOREM: The line integral of the tangential component of $\vec{F}$ taken along the boundary C of the surface S once in the positive direction equals the surface integral of the normal component of the curl of $\vec{F}$ over S.

$$\oint \vec{F}\cdot\vec{T}\,ds = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma$$

EXAMPLE 1 (Example 3 of Surface Integrals revisited) We return to our example of the paraboloid where $\vec{F}(x,y,z)=\langle y^2,x,-xz\rangle$ and compute the line integral around the boundary $C=\{(x,y,z):x^2+y^2=9,\ z=0\}$. Parametrize C by $\vec{\alpha}(t)=\langle 3\cos t,3\sin t,0\rangle$ on the interval $[0,2\pi]$.

$$\begin{aligned}
\vec{F}\big(\vec{\alpha}(t)\big) &= \langle 9\sin^2 t, 3\cos t, 0\rangle\\
\vec{\alpha}'(t) &= \langle -3\sin t, 3\cos t, 0\rangle\\
\vec{F}\big(\vec{\alpha}(t)\big)\cdot\vec{\alpha}'(t) &= -27\sin^3 t + 9\cos^2 t\\
\oint_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} -27\sin^3 t + 9\cos^2 t\,dt\\
&= \int_0^{2\pi} -27(1-\cos^2 t)\sin t + \frac{9}{2}(1+\cos 2t)\,dt\\
&= 27\cos t - \frac{27}{3}\cos^3 t + \frac{9}{2}t + \frac{9}{4}\sin 2t\Big|_0^{2\pi}\\
&= 9\pi
\end{aligned}$$



AS IT SHOULD according to Stoke's Theorem.

BUT WAIT! THERE'S MORE! C is also the boundary of the disk $D=\{(x,y):x^2+y^2\le 9, z=0\}$, so Stoke's Theorem applies there. NOTE: On D, $\vec{n}=\vec{k}$ and $(\nabla\times\vec{F})(x,y)=\langle 0,0,1-2y\rangle$ because $z=0$. So,

$$(\nabla\times\vec{F})\cdot\vec{n} = 1-2y$$

and we get the integrals

$$\begin{aligned}
\iint_D 1-2y\,dx\,dy &= \int_0^{2\pi}\int_0^3 (1-2r\sin\theta r\,dr\,d\theta\\
&= \int_0^{2\pi} \frac{r^2}{2} - \frac{2r^3}{3}\sin\theta\Big|_0^3 d\theta\\
&= \int_0^{2\pi} \frac{9}{2} - 18\sin\theta\,d\theta\\
&= 9\pi
\end{aligned}$$

OBSERVE: Stoke's Theorem in the plane is Green's Theorem. When we have a vector field $\vec{F}(x,y)=\langle M,N\rangle$ (think $\vec{F}(x,y,z)=\langle M,N,0\rangle$) and we compute the curl of $\vec{F}$ we get $\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\vec{k}$. But $\vec{n}=\vec{k}$ so

$$(\nabla\times\vec{F})\cdot\vec{n} = \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\vec{k}\cdot\vec{k} = \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$$

And we conclude $D=S$ and

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_D \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\,dx\,dy$$

as we have in Green's Theorem. The line integral

$$\oint_C \vec{F} \cdot \vec{T}\, ds$$

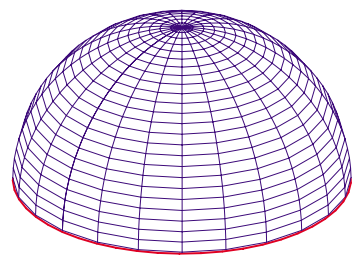
computes the <u>circulation</u> around the path C while

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

provides the accumulative effect of the curl of $\vec{F}$ over the entire region D or S.

EXAMPLE 2 Suppose we are given the curl of $\vec{F}$, $\nabla \times \vec{F} = \langle 2y, -2z, 3 \rangle$, but not the function $\vec{F}$ itself. Our surface is the upper hemisphere of radius 3. That is, $S = \{(x, y, z) : x^2 + y^2 + z^2 = 9, z \geq 0\}$. Our objective is to evaluate the surface integral $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$.

(a) Use spherical coordinates to parametrize S



$$g(\theta, \varphi) = \begin{pmatrix} 3 \sin\varphi \cos\theta \\ 3 \sin\varphi \sin\theta \\ 3 \cos\varphi \end{pmatrix} \qquad \begin{array}{c} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi/2 \end{array}$$

We have

$$\vec{n} = -\left(\frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial \varphi} \right) = \frac{\partial g}{\partial \varphi} \times \frac{\partial g}{\partial \theta} = \langle 9 \sin^2\varphi \cos\theta, 9 \sin^2\varphi \sin\theta, 9 \sin\varphi \cos\varphi \rangle$$

$$(\nabla \times \vec{F})\big(g(\theta, \varphi)\big) = \langle 6 \sin\varphi \sin\theta, -6 \cos\varphi, 3 \rangle$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = 54 \sin^3\varphi \sin\theta \cos\theta - 54 \sin^2\varphi \cos\varphi \sin\theta + 27 \sin\varphi \cos\varphi$$

Evaluating $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$ we have

$$\int_0^{\pi/2} \int_0^{2\pi} 54 \sin^3\varphi \sin\theta \cos\theta - 54 \sin^2\varphi \cos\varphi \sin\theta + 27 \sin\varphi \cos\varphi\, d\theta\, d\varphi = 27\pi$$

(b) Now we observe that the boundary of S is the circle $C = \{(x, y, 0) : x^2 + y^2 = 9\}$, which just happens to be the boundary of the disk $D = \{(x, y, z) : x^2 + y^2 \leq 9, z = 0\}$. So we apply Stoke's theorem twice.

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$$

But on D, $\vec{n} = \vec{k} = \langle 0, 0, 1 \rangle$ which means that $\nabla \times \vec{F}) \cdot \vec{n} = \langle 2y, 0, 3 \rangle \cdot \langle 0, 0, 1 \rangle = 3$ And we have

$$\iint_D 3\, d\sigma = (3)\,(\text{area}(D)) = (3)(9\pi) = 27\pi$$

Note that we were able to avoid needing to know $\vec{F}$.

Example 3, Maple Verify Stoke's Theorem for $\vec{F}(x, y, z) = \langle y^2, x, -xz \rangle$ on the paraboloid $z = 25 - x^2 - y^2$, $z \geq 0$.

```
> with(student):    with(plots):    with(linalg):
> F:=[y^2,x,-x*z];
```

$$F := [y^2, x, -xz]$$

```
> curlF:=curl(F,[x,y,z]);
```

$$curlF := [0, z, 1-2y]$$

```
> g:=[r*cos(t),r*sin(t),25-r^2];
```

$$g := [r\cos(t), r\sin(t), 25-r^2]$$

```
> curlFatg:=subs(x=g[1],y=g[2],z=g[3],op(curlF));
```

$$curlFatg := [0, 25-r^2, 1-2r\sin(t)]$$

```
> G1:=diff(g,r);
```

$$G1 := [\cos(t), \sin(t), -2r]$$

```
> G2:=diff(g,t);
```

$$G2 := [-r\sin(t), r\cos(t), 0]$$

```
> n:=crossprod(G1,G2);
```

$$n := [2r^2\cos(t), 2r^2\sin(t), \cos(t)^2 r + \sin(t)^2 r]$$

```
> grand:=innerprod(curlFatg,n);
```

$$grand := 2(25-r^2)r^2\sin(t) + (1-2r\sin(t))(\cos(t)^2 r + \sin(t)^2 r)$$

```
> grand:=simplify(grand);
```

$$48r^2\sin(t) - 2r^4\sin(t) + r$$

```
> A:=Doubleint(grand,t=0..2*Pi,r=0..5);
```

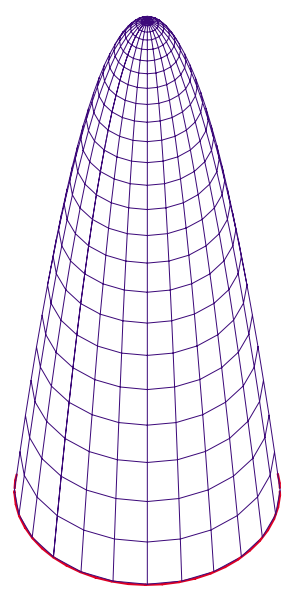
$$A := \int_0^5 \int_0^{2\pi} 48r^2\sin(t) - 2r^4\sin(t) + r\,dt\,dr$$

```
> Ans1:=value(A);
```

$$Ans1 := 25\pi$$

This is usually the more difficult answer to obtain when verifying Stoke's Theorem. Now, let's look at our surface.

```
> P1:=plot3d(g,r=0..5,t=0..2*Pi,color=blue):
> P2:=spacecurve([5*cos(t),5*sin(t),0],t=0..2*Pi,color=red):
> display(P1,P2);
```



Turning our attention to the circulation line integral, we begin by parameterizing the boundary which is shown in red in the actual plot.

```
> alpha:=[5*cos(t),5*sin(t),0];
```

$$\alpha := [5\cos(t), 5\sin(t), 0]$$

```
> Falpha:=subs(x=alpha[1],y=alpha[2],z=alpha[3],op(F));
```

$$Falpha := [25\sin(t)^2, 5\cos(t), 0]$$

```
> alphaprime:=diff(alpha,t);
```

$$alphaprime := [-5\sin(t), 5\cos(t), 0]$$

```
> grand1:=innerprod(Falpha,alphaprime);
```

$$grand1 := -125\sin(t)^3 + 25\cos(t)^2$$

```
> Lint:=Int(grand(t=0..2*Pi);
```

$$Lint := \int_0^{2\pi} -125\sin(t)^3 + 25\cos(t)^2\, dt$$

```
> Ans2:=value(Lint);
```

$$Ans2 := 25\pi$$

And we see that our answers agree, as they should.

C3M16 Problems

Verify Stoke's Theorem for:

1. $\vec{F}(x,y,z) = \langle 2z, 3x, 4y\rangle, \quad S = \{(x,y,z) : 0 \le z = 9 - x^2 - y^2\}.$

2. $\vec{F}(x,y,z) = \langle 4x^2 + y, 2y, x - 3z^2\rangle, \quad S = \{(x,y,z) : 3x + y + 3z = 4, x, y, z \ge 0\}.$

Apply Stoke's Theorem to evaluate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$:

3. $\vec{F}(x,y,z) = \langle 0, x - y, z^3\rangle \quad S$ is the surface of the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$

4. Let the surface S be the curved side of the cylinder $x^2 + y^2 = 9,\ 0 \le z \le 2$ together with the top, $x^2 + y^2 \le 9, z = 2$. $\vec{F}(x,y,z) = \langle -y, x, x^2\rangle$.

C3M16 Maple Problems Use Maple in the remaining problems.

5. Apply Stoke's Theorem so as to evaluate $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$ if $\vec{F}(x,y,z) = \langle 0, x - y, z^3\rangle$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$

6. Apply Stoke's Theorem so as to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$. C is the square with vertices $P_1(0,0,4),\ P_2(3,0,4),\ P_3(3,2,4),\ P_4(0,2,4)$ taken in that order and $\vec{F}(x,y,z) = \langle y/z, x^2y, x + z\rangle$.