C1M6

Limits

Graduating from high school in 1956 and driving cars with wide front seats and no seatbelts, one developed a sense of how things were going on a date by where she chose to sit. If she sat by the door and gripped the armrest with white knuckles, then things weren't going well. On the other hand, if she sat close, then one had to guard against grinning stupidly and spoiling the moment. "Close to" had real meaning. This is also true in mathematics whenever limits are being discussed. There are always two ingredients in the discussion and they behave as *accuracy* and *control*, and accuracy always precedes control. Let's consider the following sequence of numbers:

$x_1 = 1$	$x_1^2 = 1$
$x_2 = 1.4$	$x_2^2 = 1.96$
$x_3 = 1.41$	$x_3^2 = 1.9881$
$x_4 = 1.414$	$x_4^2 = 1.999396$
$x_5 = 1.4142$	$x_5^2 = 1.99996164$
$x_6 = 1.41421$	$x_6^2 = 1.9999899241$
$x_7 = 1.414213$	$x_7^2 = 1.999998409369$
$x_8 = 1.4142135$	$x_8^2 = 1.99999982358225$
$x_9 = 1.41421356$	$x_9^2 = 1.9999999932878736$
$x_{10} = 1.414213562$	$x_{10}^2 = 1.999999998944727844$
.	.
.	.

The right column lists the squares of the left column, and we can see that the numbers in the right column are getting 'closer' to 2. Of course, this means that the numbers in the left column are getting 'closer' to $\sqrt{2}$. It is certainly fair to say that we are *approximating* $\sqrt{2}$ by increasing our accuracy one place each time we select a new number in the left column. Suppose that we wanted to approximate $\sqrt{2}$ to within $.000001$, which we may regard as an accuracy. Certainly if we go down our list to x_9 and look at $x_9^2 = 1.9999999932878736$ and at $|(x_9)^2 - 2| = .0000000067121264$, we see that we have achieved our accuracy. But, where does control fit in here? Note first that the numbers that we would list below x_9 would be even closer to $\sqrt{2}$ than x_9 is, we control the situation by selecting a point on our list where our accuracy is achieved for all the rest of the list. But, our focus will be on limits of functions rather than on sequences.

Let's assume that we have a laser attached to a rifle and that it is adjusted so that when the rifle is fired at a target, the bullet will strike exactly where the laser points at that instant. But, the person firing the weapon is not perfectly rigid and the aim can vary. Suppose further that the "bull's-eye" is ten inches across and we wish to hit the bull's-eye. In effect, we have selected an accuracy of 5 inches. To make things simpler, assume that the butt of the rifle is set on a fixed point and that we can measure the deviation of the barrel tip from 'perfect'. The question becomes, "What is a permissible deviation from perfect that ensures that the bullet will strike the bull's-eye?" This is our control. Suppose that when the deviation is less than .0357 inches from perfect, we are assured that the bull's-eye will be struck. For an accuracy of 5 inches we found a control of .0357 inches that would guarantee that the accuracy would be achieved. It would seem that for each accuracy selected, we could find a permissible deviation (control) that would guarantee that the bullet would strike the target within the chosen accuracy of the center of the bull's-eye. And this is how limits work.

Definition: (Limit of a function) We write

$$\lim_{x \to a} f(x) = L$$

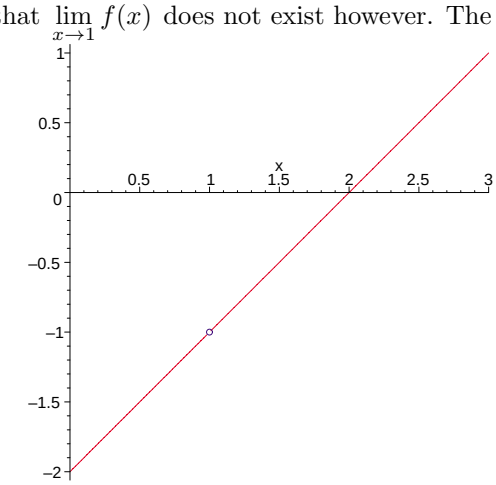
if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $x \neq a$ and $|x - a| < \delta$, then it follows that $|f(x) - L| < \epsilon$.

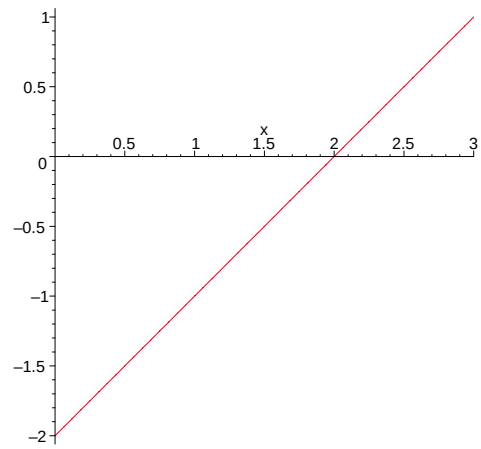
This is just a precise way of saying that the values of $f(x)$ are as close to L as we like whenever x is close enough to a, but is not equal to a. We call L 'the limit of $f(x)$ as x approaches a'.

Reminder: $|x-a|<\delta \iff a-\delta < x < a+\delta$

It is important now to comment on how when we are discussing a limit at $x=a$, we actually ignore the value, if any, of the function at a. In fact, we frequently look at only what is happening when $x<a$ (left-hand limit) or at when $x>a$ (right-hand limit). Then we compare our answers.

Suppose that $f(x)=\dfrac{x^2-3x+2}{x-1}$. We see immediately that f is not defined at $x=1$. This does not mean that $\lim\limits_{x\to 1} f(x)$ does not exist however. The graph is displayed on the left below.





The Maple code shown below produced the plot on the right above, in addition to the output that is displayed.

```
> restart:
> f:=x->(x^2-3*x+2)/(x-1);
```

$$f := x \to \frac{x^2-3x+2}{x-1}$$

```
> plot(f(x),x=0..3,scaling=constrained);
```

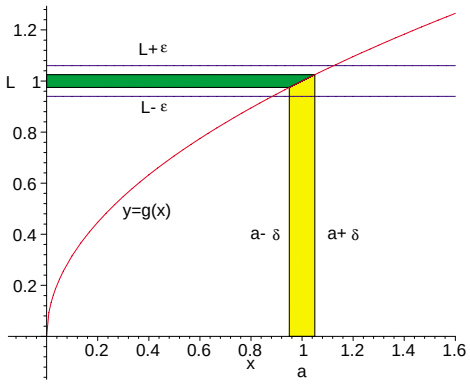
Output above on right

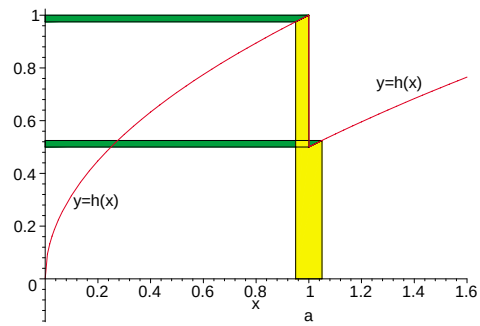
```
> Limit(f(x),x=1)=limit(f(x),x=1);
```

$$\lim_{x\to 1} \frac{x^2-3x+2}{x-1} = -1$$

What should we learn from this? Maple may not show a 'hole' in a graph, so we must not depend on Maple to show us problem points. When taking a limit in Maple and a capital 'L' is used, the expression is inert. That is, the operation is not executed. On the other hand, the lower case 'l' allows Maple to complete taking the limit.

Graphical discussion: The objective here is to show a picture of how a limit works by examining a graph of two functions, g and h, demonstrating the process. First, there is an"x" value a at which the limit is to be discussed. Then by some mysterious process we select the limit L and any accuracy $e>0$. Now we draw two symmetric horizontal lines, $y=L-e$ and $y=L+e$ which determine our bounds. Then we consider two symmetric vertical lines about $x=a$ and all the points between, except a itself. The question is, "Can we move the lines together by making the number d smaller, but still positive, and have the functional values project up to the graph and over to the y-axis in such a way that the projected values remain between the two horizontal lines?" In the first diagram below the answer is "yes", and in the second the answer is "no". In fact, there is not even a candidate for L in the second one because of the manner in which the values of $h(x)$ are split.





In the case of $g(x)$ above, no matter how small we make e, we can always find a d small enough so that the diagram above on the left is valid. This means that $\lim_{x \to a} g(x) = L$. The function $h(x)$, as drawn, has no limit at $x = a$.

We have mentioned one-sided limits, and now it is time to be specific about them. The function h above will serve as our mental picture. What happens when we restrict x so that $a - d < x < a$ (left-hand limit), or so that $a < x < a + d$ (right-hand limit)? As x approaches a from the left-hand side we sense that $h(x)$ is getting close to a number which we will call L_1. Also, as x approaches a from the right-hand side $h(x)$ is getting close to a number L_2. It is obvious that L_1 and L_2 are not the same numbers in this case. For completeness, we will provide precise definitions of one-sided limits.

Definition: (Left-hand limit of a function) We write

$$\lim_{x \to a^-} f(x) = L_1$$

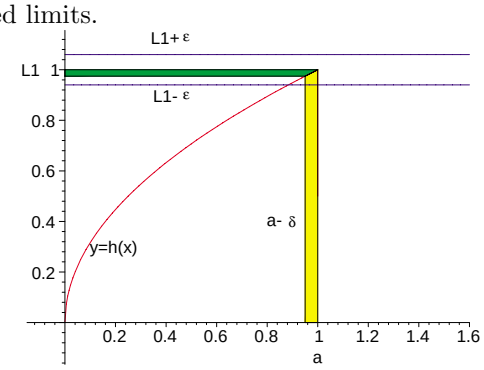
if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $a - \delta < x < a$, then it follows that $|f(x) - L_1| < \epsilon$.

Definition: (Right-hand limit of a function) We write

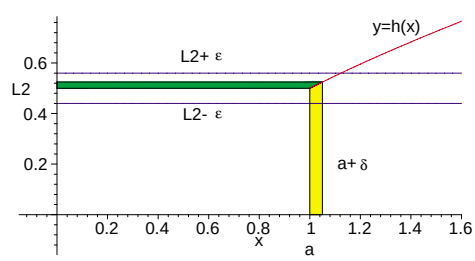
$$\lim_{x \to a^+} f(x) = L_2$$

if for each $\epsilon > 0$ (accuracy) there is a number $\delta > 0$ (control), so that whenever $a < x < a + \delta$, then it follows that $|f(x) - L_2| < \epsilon$.

Here is the graphical picture of what is happening. If we select a smaller accuracy ($e > 0$), then we may select a smaller control ($d > 0$), so that the functional values of those x's lie within the requested range close to the limit. Note that we only provided one half of the graph of $h(x)$ in each case. This is to emphasize that only what happens on the left and right sides respectively of $x = a$ matters when determining the one-sided limits.



$\lim_{x \to a^-} h(x) = L_1$



$\lim_{x \to a^+} h(x) = L_2$

It should be obvious that there is a relationship between limits and one-sided limits. We usually tie this up nicely with a theorem that shows the equivalence of two statements. But, it is a little easier for the student to understand if it is stated as two related theorems and a corollary, and that is what we will do here.

Theorem A:

$$\lim_{x\to a} f(x) = L \quad \Longrightarrow \quad \lim_{x\to a^-} f(x) = L \ \textbf{and} \ \lim_{x\to a^+} f(x) = L$$

Theorem B:

$$\lim_{x\to a^-} f(x) = L_1 \text{ and } \lim_{x\to a^+} f(x) = L_2 \ \textbf{and} \ L_1 = L_2 \quad \Longrightarrow \quad \lim_{x\to a} f(x) = L_1 \ (= L_2)$$

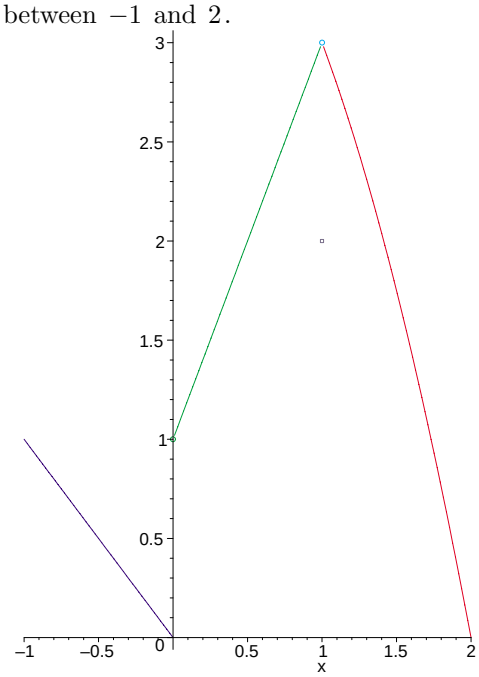
Corollary:

$$\lim_{x\to a^-} f(x) = L_1 \text{ and } \lim_{x\to a^+} f(x) = L_2 \ \textbf{and} \ L_1 \neq L_2 \quad \Longrightarrow \quad \lim_{x\to a} f(x) \ \textbf{does NOT exist}$$

Maple Example: A function f may be defined by

$$f(x) = \begin{cases} -x, & \text{if } -1 \le x \le 0; \\ 2x+1, & \text{if } 0 < x < 1; \\ 2, & \text{if } x = 1; \\ 4 - x^2, & \text{if } 1 < x \le 2. \end{cases}$$

Here is a look at the graph of f between -1 and 2.



Suppose we want to find if f has a limit at $x = 0$ and $x = 1$. We look at the graph for inspiration and realize that if we stay to the left of $x = 0$, then the function is tending to 0 as x gets closer to the value 0. Also, if we stay to the right of $x = 0$ and allow x to decrease towards 0, then the values of f are getting close to 1. It would seem that the left-hand limit is 0 and the right-hand limit is 1. Then, if we take the same approach at $x = 1$, but we must ignore $x = 1$, and allow x to get closer and closer to 1, then $f(x)$ will get closer to 3, no matter which side of 1 x happens to lie on. All this suggests intuitively that

$$\lim_{x\to 0^-} f(x) = 0, \ \lim_{x\to 0^+} f(x) = 1 \ \Longrightarrow \ \lim_{x\to 0} f(x) \text{ does not exist}$$

$$\lim_{x\to 1} f(x) = 3$$

Now we turn to Maple.

```
> restart:
> f:=x->piecewise(x>=-1 and x<=0,-x,x<1,2*x+1,x=1,2,x>1 and x<=2,4-x^2);
```

$$f := x \to piecewise(-1 \le x \ \textbf{and} \ x \le 0, -x, x < 1, 2x+1, x = 1, 2, 1 < x \ \textbf{and} \ x \le 2, 4 - x^2);$$

```
> limit(f(x),x=0,left);
```

$$0$$

```
> limit(f(x),x=0,right);
```

$$1$$

```
> limit(f(x),x=0);
```

$$undefined$$

```
> limit(f(x),x=1);
```

$$3$$

```
> f(-1/2); f(1/2);  f(1);
```

$$\frac{1}{2}$$

$$2$$

$$2$$

You see above an example of how to define a function piecewise. You list the conditions on x first, and then the function's values for those x's. When there are two conditions on x, you must separate them with 'and'. You see how easy it is to obtain regular and one-sided limits.

C1M6 Problems: Use Maple to plot the graphs and to find the limits at the indicated points, if they exist.

1. $f(x) = \dfrac{\sin(2x)}{3x}$, $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$, at $x = 0$.

2. $g(x) = \sin\left(\dfrac{1}{x}\right)$, $-\pi \le x \le \pi$, at $x = 0$.

3. $h(x) = x\,\sin\left(\dfrac{2}{x}\right)$, $-\pi \le x \le \pi$, at $x = 0$.

4. $F(x) = \dfrac{\sin(x) - x)}{x^3}$, $-\pi \le x \le \pi$, at $x = 0$.

5. $G(x) = \begin{cases} x+1, & \text{if } -1 \le x \le 0; \\ x^2, & \text{if } 0 < x \le 2; \\ 6-x, & \text{if } 2 \le x \le 4. \end{cases}$, at $x = 0$ and $x = 2$.