CALCULUS II FINAL EXAM 0750-1050 TUESDAY, 8 MAY 2001
(SM122, SM122A, SM122N) PAGE 1 OF 4 PAGES

You should have a calculator. Write your name, alpha number, and section on your blue book(s) and the bubble sheet. Bubble in your alpha number in the left-most columns of the bubble sheet.

Part One. Multiple choice (50%). The first 20 problems are multiple choice. Fill in the letter of the best answer on your bubble sheet. There is no penalty for a wrong answer. YOU MUST ALSO WRITE YOUR ANSWER AND SHOW ALL YOUR WORK IN YOUR BLUE BOOK(S).

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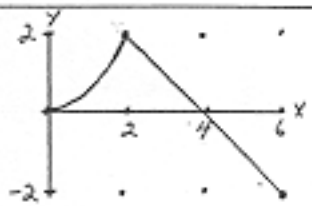
1. Determine $\int_0^6 f(x)\,dx$ for the function f whose graph on the right consists of the curve $y = x^3/4$ and a line segment.



a) 0 b) 1 c) 5 d) 8 e) 20

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2. If $\int_0^1 g(t)dt = 2$, $\int_0^4 g(t)dt = -6$, and $\int_3^4 g(t)dt = 1$, then $\int_1^3 g(t)dt =$

a) -9 b) -3 c) 3 d) 9 e) 12

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3. If $\int_1^2 h(x)dx = 5$, then $\int_1^2 (4 - 2x + 3h(x))\,dx =$

a) 4 b) 8 c) 15 d) 16 e) 24

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4. The substitution $u = \sqrt{x}$ changes the definite integral $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ to:

a) $\int_1^2 e^u du$ b) $2\int_1^2 e^u du$ c) $2\int_1^4 e^u du$ d) $\int_1^4 e^u du$ e) $\int_1^2 e^{\sqrt{u}} du$

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5. One application of integration by parts reduces $\int [\ln(x)]^2 dx$ to:

a) $2[\ln(x)]/x - 2\int \ln(x)\,dx$ b) $[\ln(x)]^2 - 2\int \ln(x)\,x\,dx$

c) $[\ln(x)]^2 x - 2\int \ln(x)\,dx$ d) $1/x^2 - \int (1/x)\,dx$ e) $(1/x^2)\ln(x) - \int x\,dx$

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6. Evaluate the improper integral $\int_1^\infty f(x)\,dx$ given that $\int_1^t f(x)\,dx = 2 - \frac{2}{\sqrt{t}}$.

a) 0 b) 1 c) 2 d) 3 e) The integral diverges.

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7. Oil leaks from a tank at the rate of $3(t-20)^2$ gallons per minute. How many gallons leak from the tank from time $t = 10$ to $t = 20$ minutes ?

a) 0 b) 100 c) 500 d) 1000 e) 10000

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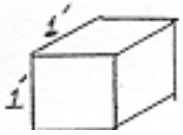
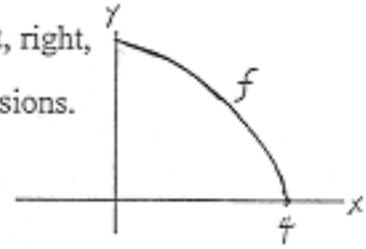
8. The work done is pumping the water out of the top of a full aquarium with dimensions 1ft x 1ft x 1ft is closest to: (Water weighs 62.5 lbs/ft^3).



a) 16 ft-lbs b) 21 ft-lbs c) 31 ft-lbs d) 62 ft-lbs e) 125 ft-lbs

9. For the function graphed on the right, let L_n, R_n, M_n, T_n represent the left, right, midpoint, and trapezoidal rule approximations to $\int_0^4 f(x)\,dx$ using n subdivisions. For any value of n, list the numbers L_n, R_n, M_n, T_n in increasing order.



a) $L_n < R_n < M_n < T_n$ b) $L_n < M_n < R_n < T_n$
c) $L_n < M_n < T_n < R_n$ d) $R_n < M_n < T_n < L_n$ e) $R_n < T_n < M_n < L_n$

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10. What is the distance between the point P whose polar coordinates are $(r,\theta) = (2,\pi/3)$ and the point Q whose rectangular coordinates are $(x,y) = (-2,\sqrt{3})$?

a) $\sqrt{16+(\pi/3-\sqrt{3})^2}$ b) 3 c) 4 d) $4+\pi/3+\sqrt{3}$ e) 5

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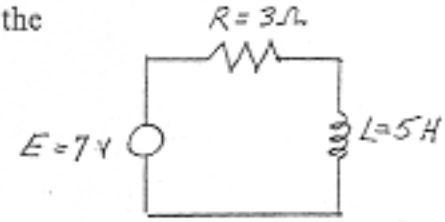
11. According to Kirchoff's Law, the current $I(t)$ running through the closed electric circuit on the right satisfies the differential equation:



a) $5\frac{dI}{dt}+3I=7$ b) $3\frac{dI}{dt}+\frac{1}{5}I=7$ c) $3\frac{dI}{dt}+5I=7$
d) $7\frac{dI}{dt}+\frac{5}{I}=3$ e) $5\frac{dI}{dt}+7I=3$

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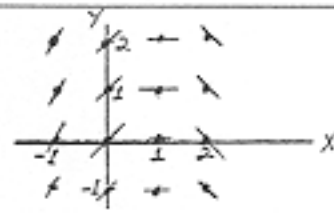
12. Approximate $y(2)$ using two steps of Euler's method with step size $h=1$, if $y(0)=0$ and y satifies a first order differential equation whose direction field is shown on the right.



a) -1 b) 0 c) 1 d) 2 e) 3

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13. Use separation of variables to solve $y' = f(x)y$ if $\int f(x)\,dx = \cos(x^2)+C$.

a) $y = Ae^{\cos(x^2)}$ b) $y = (\cos(x^2)+C)\frac{y^2}{2}$ c) $y = e^{\cos(x^2)}+e^C$ d) $y = -\sin(x^2)2x$ e) $y = Ae^{\sin(x^2)}$

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14. The infinite geometric series $2004 + 668 + \ldots$ converges to the sum:

a) 2004 b) 2672 c) 3006 d) 4008 e) ∞

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15. The radius of convergence for the power series $\sum_{n=1}^{\infty}\frac{n(x-3)^n}{4^n}$ is:

a) 0 b) 3 c) ∞ d) 1/4 e) 4

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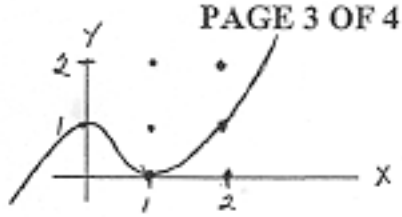
16. If $f(1)=2$, $f'(1)=3$, *and* $f''(1)=4$, use the 2nd degree Taylor polynomial center at a = 1 for for f to approximate $f(2)$.

a) 7 b) 2 c) 3 d) 4 e) 9

17. Which of the following could be a Taylor series for the function graphed on the right?



a) $2+(x-2)+(x-2)^2+\ldots$ b) $1-(x-2)-(x-2)^2+\ldots$
c) $(x-2)+(x-2)^2+\ldots$ d) $1+(x-2)-(x-2)^2+\ldots$ e) $1+(x-2)+(x-2)^2+\ldots$

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18. Each of the following is a vector, a scalar, or makes no sense. Which is a vector?
a) $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ b) $(\vec{a}\times\vec{b})\cdot\vec{c}$ c) $(\vec{a}\cdot\vec{b})/(\vec{c}\times\vec{d})$ d) $(\vec{a}\cdot\vec{b})\times\vec{c}$ e) none of these

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19. Find the scalar component $(comp_{\vec{a}}\,\vec{b})$ of the vector $\vec{b}$ in the direction of $\vec{a}$ if $\vec{b}=<1,0,5>$ and $\vec{a}=<0,3,4>$.
a) 4 b) $<0,6,8>$ c) 12 d) 5 e) 3

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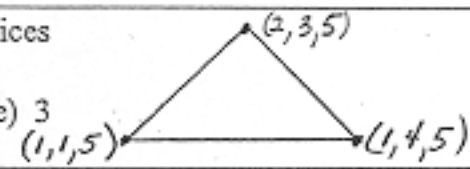
20. Use vectors to find the area of the triangle whose vertices are given in the graph to the right.



a) 1 b) 1.5 c) 2 d) 2.5 e) 3

Part Two. Longer Answers (50%). **SOLVE ANY 10 OF THE REMAINING 11 PROBLEMS.** They are not multiple choice. Show all of your work and put your answers in your blue book(s).

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21. Evaluate the following integrals showing all of your steps as if you do not have a calculator:
a) $\int \sec^2(2004t)\,dt$ b) $\int x\cos(x)\,dx$ c) $\int r e^{r^2}\,dr$

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22. a) sketch a graph showing the region R bounded by the curves $x=y^2$ and $y=x-2$.
b) Find the area of the region R.

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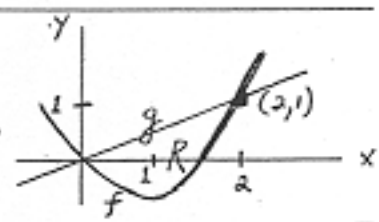
23. If R is the region graphed on the right bounded by the curves $y=f(x)$ and $y=g(x)$, <u>set up</u> integrals to find:
a) the volume generated by revolving R about the horizontal line $y=1$,
b) the volume generated by revolving R about the y axis.



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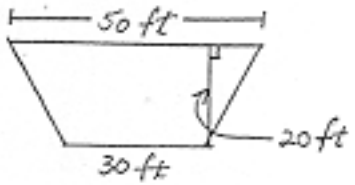
24. Use Simpson's rule with n = 4 to approximate the length of the curve given parametrically by $x=f(t)$, $y=g(t)$ for $t=0$ *to* $t=8$ given the information in the table to the right.

t	0	2	4	6	8
$f'(t)$	4	2	0	1	3
$g'(t)$	3	$\sqrt{5}$	1	$\sqrt{3}$	0

Recall Simpson's rule says that $\int_a^b h(t)\,dt$ is approximately

$(\Delta t/3)[h(t_0)+4h(t_1)+2h(t_2)+\ldots+2h(t_{n-2})+4h(t_{n-1})+h(t_n)]$.

25. A dam has the shape of the trapezoid shown on the right. Find the force on the dam due to hydrostatic pressure if the dam is filled to the top with water. Use the fact that the hydrostatic pressure at a depth of x feet is $(62.5)(x)\text{lbs/ft}^2$.



26. If $y(t)$ is the temperature of an object at time t (minutes) in a room of temperature R, then $y(t)$ satisfies the differential equation $\frac{dy}{dt} = k(y-R)$.

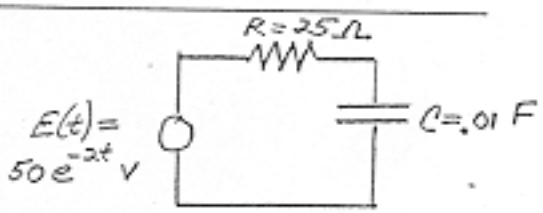
a) Explain in plain English what this equation means.

b) A cup of coffee has temperature 150° F when it is taken out of a microwave (t = 0) and placed in a room of temperature 70° F. Ten minutes later the coffee has temperature 110° F. Set up and solve a differential equation using the separation of variable method to find the temperature of the coffee at any time t.

27. a) State Kirchoff's Law.

b) Set up a differential equation for the charge $Q(t)$ on the capacitor in the circuit to the right.

c) Solve the differential equation for the charge $Q(t)$ using the integrating factor method if at $t = 0$, $Q = 4$ coulombs.



28. Consider the differential equation $y' = x - y$.

a) Sketch its direction field.

b) Graph the solution satisfying $y(0) = -1$ on your direction field from part a).

c) Find the solution satisfying $y(0) = -1$. (Hint: You can write the equation for your curve in part b) and verify that it is a solution, or you can solve the differential equation.)

29. For each of the following infinite series, state whether it converges or diverges. Show clearly the test that you used to determine your answer:

a) $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{n+1}$ b) $\sum_{n=1}^{\infty} \frac{n!}{n\,2^n}$

30. a) Use the definition for the Maclaurin series for a function $f(x)$ to find the Maclaurin series for $\sin(x)$. Write out the first three non-zero terms and the summation notation for the series.

b) Use your result from part a) to approximate $\int_0^{1/2} \sin(x^2)\,dx$ to within an accuracy of .000001.

31. a) Sketch the graph of the curve given in polar coordinates by $r = \sin(3\theta)$ for $\theta = 0$ to 2π.

b) Find the area enclosed by the curve in part a).

Calculus II Final Exam Solutions 8 May 2001

1. $\int_0^6 f(x)\,dx = \int_0^2 \frac{x^3}{4}\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx$

$= \frac{x^4}{16}\Big|_0^2 + 2 - 2 = \frac{2^4}{16} = 1$ (b)

2. $\int_1^3 g(t)\,dt = \int_0^4 g(t)\,dt - \int_0^1 g(t)\,dt - \int_3^4 g(t)\,dt$

$= (-6) - (2) - (1) = -9$ (a)

3. $\int_1^2 (4 - 2x + 3h(x))\,dx$

$= (4x - x^2)\Big|_1^2 + 3\int_1^2 h(x)\,dx$

$= (8-4) - (4-1) + 3(5) = 16$ (d)

4. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ let $u = \sqrt{x} = x^{1/2}$.

$\Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$= 2\int_1^2 e^u\,du$ $\Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$

$1 \xrightarrow{x} 4$

$1 \xrightarrow{u} 2$

(b)

5. $\int [\ln(x)]^2\,dx$ let $u = [\ln(x)]^2$

$\Rightarrow du = 2\ln(x)\cdot\frac{1}{x}\,dx$

$dv = dx \Rightarrow v = x$

(c) $= [\ln(x)]^2 x - \int \cancel{x}\cdot 2\ln(x)\cdot\frac{1}{\cancel{x}}\,dx$

$(= uv - \int v\,du)$

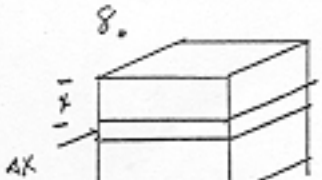
6. $\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\left[\int_1^t f(x)\,dx\right]$

$= \lim_{t\to\infty}\left[2 - 2/\sqrt{t}\right] = 2$ (c)

7. $\int_{10}^{20} 3(t-20)^2\,dt = (t-20)^3\Big|_{10}^{20}$

$= 0 - (-10)^3 = 1000$ gals (d)

8.



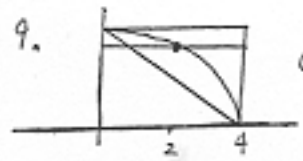
$\Delta W = \Delta F \cdot d$

$= 62.5\frac{lbs}{ft^3}\cdot \Delta V\,ft^3 \cdot x\,ft$

$= 62.5\frac{lbs}{ft^3}\cdot(\Delta x(1)(1))\,ft^2 \cdot x\,ft$

$W = \int_0^1 62.5x\,dx = 62.5\frac{x^2}{2}\Big|_0^1 = 31.25$ ft-lbs (c)

9.



(e) $R_n < T_n < M_n < L_n$

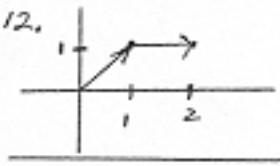
10. P: $x = r\cos(\theta) = 2\cos(\pi/3) = 1$

$y = r\sin(\theta) = 2\sin(\pi/3) = \sqrt{3}$

$d(P,Q) = \sqrt{[1-(-2)]^2 + [\sqrt{3} - \sqrt{3}]^2} = 3$ (b)

11. $E_R + E_L = EMF \Rightarrow 3I + 5I' = 7$ (a)

12.



$y(2) \doteq 1$ (c)

or $y_1 = y_0 + h\,y'(x_0)$
$= 0 + 1(0.1) = 1$
$y_2 = y_1 + h\,y'(t_1)$
$= 1 + 1\cdot 0 = 1$

13. $\frac{dy}{dx} = f(x)y \Rightarrow \int\frac{1}{y}\,dy = \int f(x)\,dx$

$\Rightarrow e^{\ln|y|} = e^{[\cos(x^2)+C]} \Rightarrow |y| = e^{\cos(x^2)}\cdot e^C$

$\Rightarrow y = Ae^{\cos(x^2)}$ (a)

14. $a = 2004$, $r = \frac{1}{3}$. $\frac{a}{1-r} = \frac{2004}{2/3} = 3006$ (c)

15. (ratio test) $L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left|\frac{(x+1)(x-3)^{n+1}}{4^{n+1}}\right|\left|\frac{4^n}{n(x-3)^n}\right|$

$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\frac{|x-3|}{4} = \frac{|x-3|}{4} < 1 \Rightarrow |x-3| < 4 = R$ (e)

16. $f(x) \doteq f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$

so $f(2) \doteq 2 + 3(2-1) + \frac{4}{2}(2-1)^2 = 7$ (a)

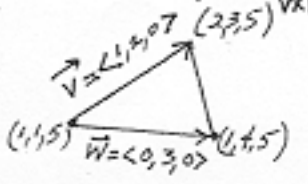
17. $f(2) = 1$, $f'(2) > 0$, $f''(2) > 0$

$1 + 1(x-2) + 1(x-2)^2 + \dots$ (e)

18. (e) none of these

19. $\text{comp}_{\vec{a}}\vec{b} = \vec{b}\cdot\frac{\vec{a}}{|\vec{a}|} = \langle 1,0,5\rangle\cdot\frac{\langle 0,3,4\rangle}{\sqrt{9+16}} = \frac{0+0+20}{5} = 4$ (a)

20. area of $\Delta = \frac{1}{2}|\vec{v}\times\vec{w}|$.



$\vec{v}\times\vec{w} = \langle 1,2,0\rangle \times \langle 0,3,0\rangle$

$= \langle 0,0,3\rangle$.

$\frac{1}{2}|\langle 0,0,3\rangle| = \frac{3}{2}$ (b)

21. a) $\int \sec^2(\underbrace{2004t}_{u})\,dt = \frac{1}{2004}\int \sec^2(u)\,du$

$= \boxed{\frac{1}{2004}\tan(2004t) + C}$

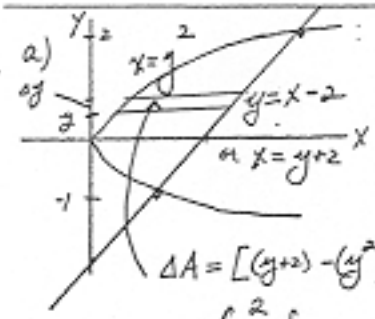
b) $\int \underbrace{x}_{u}\,\underbrace{\cos(x)\,dx}_{dv} = \underbrace{x}_{u}\underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v}\,\underbrace{dx}_{du}$

$= \boxed{x\sin(x) + \cos(x) + C}$

c) $\int r e^{r^2}\,dr$ let $u = r^2$, $du = 2r\,dr$

$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{r^2} + C}$

22. a)



$y^2 = y+2$

$y^2 - y - 2 = 0$

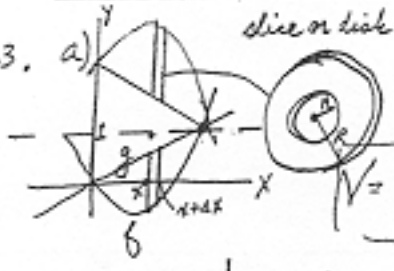
$(y-2)(y+1) = 0$

$y = -1, 2$

$\Delta A = [(y+2) - (y^2)]\,\Delta y$

b) $A = \int_{-1}^{2} [(y+2) - (y^2)]\,dy = \boxed{\frac{9}{2}}$

23. a)



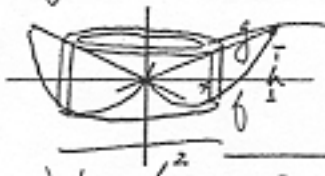
disc or disk or washer

$\Delta V = \pi(R^2 - r^2)\,\Delta x$

$= \pi\left([1-f(x)]^2 - [1-g(x)]^2\right)\Delta x$

$\boxed{V = \int_0^2 \pi\left([1-f(x)]^2 - [1-g(x)]^2\right)dx}$

(b)



cylindrical shell

$\Delta V = 2\pi x\,h\,\Delta x$

$= 2\pi(x)(g(x) - f(x))\,\Delta x$

so $\boxed{V = \int_0^2 2\pi x\,[g(x) - f(x)]\,dx}$

24. $l = \int_0^8 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

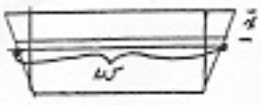
$= \int_0^8 \underbrace{\sqrt{[f'(t)]^2 + [g'(t)]^2}}_{h(t)}\,dt$

$\doteq \frac{2}{3}[h(0) + 4h(2) + 2h(4) + 4h(6) + h(8)]$

$= \frac{2}{3}[5 + 4(3) + 2(1) + 4(2) + 3]$

$= \frac{2}{3}[5 + 12 + 2 + 8 + 3] = \frac{2}{3}[30] = \boxed{20}$

25.



$\Delta F = \text{pressure} \cdot \Delta\text{area}$

$= \left(62.5x\,\frac{\text{lbs}}{\text{ft}^3}\right)\cdot(\Delta x \cdot w)\ \text{ft}^2$

top

$\frac{l}{10} = \frac{20-x}{20} \Rightarrow l = 10 - \frac{1}{2}x$

$w = 30 + 2l = 30 + 20 - x = 50 - x$

$F = \int_0^{20} 62.5x \cdot (50 - x)\,dx = 458{,}333.\overline{3}$ lbs.

26. a) At any moment, the rate of change of the temperature of the object is proportional to the temperature difference of the object and the room.

(b) $\frac{dy}{dt} = k(y - 70) \Rightarrow \int \frac{1}{y-70}\,dy = \int k\,dt$

$\Rightarrow \ln|y - 70| = kt + C \Rightarrow y - 70 = Ae^{kt} + 70$

$y(0) = 150 = Ae^0 + 70 \Rightarrow A = 80 \Rightarrow y = 80e^{kt} + 70$

$y(10) = 110 = 80e^{10k} + 70 \Rightarrow \frac{1}{2} = e^{10k}$

$\Rightarrow k = \frac{1}{10}\ln\left(\frac{1}{2}\right) = -.0693$

so $\boxed{y = 80e^{-.0693t} + 70}$

27. (a) The sum of the voltage drops around a closed circuit = voltage supplied.

(b) $25I + \frac{1}{.01}Q = 50e^{-2t}$ or

$Q' + 4Q = 2e^{-2t}$

(c) integ. factor $= e^{\int 4dt} = e^{4t}$

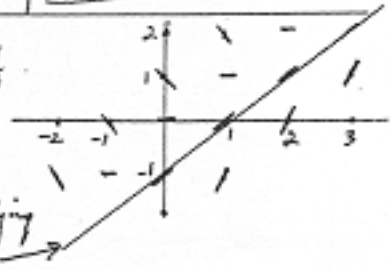
$e^{4t}[Q' + 4Q] = e^{4t}(2e^{-2t})$

$[e^{4t}Q]' = 2e^{2t} \Rightarrow e^{4t}Q = e^{2t} + C$

$\Rightarrow Q = e^{-2t} + Ce^{-4t}$. $Q(0) = 4 \Rightarrow 4 = 1 + C$

$\Rightarrow C = 3$ so $\boxed{Q = e^{-2t} + 3e^{-4t}}$

28. a) $y' = x - y$



b) sol'n satisfying $y(0) = -1$

c) looks like $y = x - 1$

check $(x-1)' = \cancel{x} - (\cancel{x} - 1)$

$1 = 1$ ✓

Calculus II Final Exam Solutions 8 May 2001

29. (a) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n}{n+1} = 2 \neq 0$

diverges by the divergence test.

(b) ratio test: $L = \lim \frac{|a_{n+1}|}{|a_n|}$

$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{(n+1)\cancel{2^{n+1}}_{\,2}} \cdot \frac{n\cancel{2^n}}{\cancel{n!}}$

$= \lim_{n\to\infty} \frac{n}{2} = \infty > 1$ so

the series diverges.

30. $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$

$f(x) = \sin(x)\big|_{x=0} = 0$

$f'(x) = \cos(x)\big|_{x=0} = 1$

$f''(x) = -\sin x\big|_{x=0} = 0$

$f'''(x) = -\cos(x)\big|_{x=0} = -1$

$\vdots \qquad \vdots$

$\Rightarrow \sin(x) = 0 + x + 0x^2 - \frac{1x^3}{3!} + \cdots$

a) or $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$

b) $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$

$\int_0^{.5} \sin(x^2)\,dx = \frac{x^3}{3} - \frac{x^7}{7\cdot 6} + \frac{x^{11}}{11\cdot 120} - \cdots \Big|_0^{.5}$

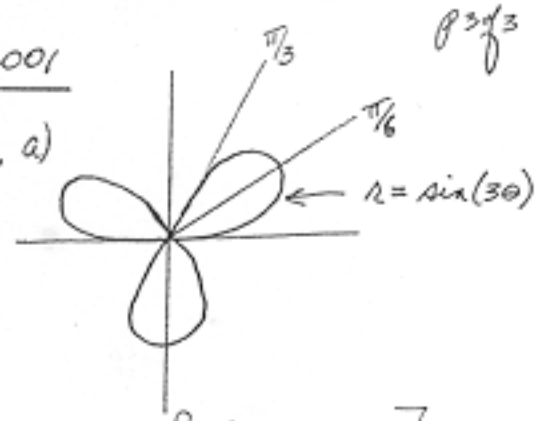
$= \frac{.5^3}{3} - \frac{.5^7}{42} + \frac{.5^{11}}{1320} - \cdots$ alt, decreasing

$\doteq \underbrace{.041666\overline{6} - .000\,186\,012}\Big| + .000\,000\,370$

$< .000\,001$

$\doteq .041\,481$

31. a)



b) area $= 6\left[\int_0^{\pi/6} \frac{1}{2} r^2\, d\theta\right]$

$= 3\int_0^{\pi/6} \sin^2(3\theta)\, d\theta$

$= \pi/4$ [calculator]

or $3\int_0^{\pi/6} \frac{1-\cos(6\theta)}{2}\, d\theta$

$= \frac{3}{2}\left[\theta - \frac{\sin(6\theta)}{6}\right]\Big|_0^{\pi/6}$

$= \frac{3}{2}\left[\pi/6 - 0\right] = \pi/4$