C2 NOTES

A Calculus and MAPLE V Supplement

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by Professor C. E. Moore

Review of Differentiation and Anti-differentiation

As we begin Calculus II it is assumed that the student has covered the derivative of a function in some detail and has learned how to find the anti-derivative of basic functions. This section is provided as a summary of some of these topics. Knowledge of trigonometric functions, exponential functions, and logarithmic functions is assumed, but will be reviewed as necessary throughout the course for reinforcement.

Definition of derivative. Suppose f is defined on an open interval containing x. The derivative of f at x is defined by

$$D_x\left(f(x)\right) = f'(x) \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Tangent line. If $f(x_0) = y_0$ and $f'(x_0) = m$, then an equation for the line tangent to the curve $y = f(x)$ is given by

$$y - y_0 = m(x - x_0)$$

Rules of Differentiation Assume that a and b are real numbers and that $f(x)$ and $g(x)$ are differentiable on an open interval containing x.

Rule 1. The derivative is linear. That is, $D_x\big(af(x)+bg(x)\big) = aD_x(f(x)) + bD_x(g(x)) = af'(x) + bg'(x)$.

Rule 2. Product Rule. $D_x\big(f(x)g(x)\big) = D_x(f(x))g(x) + f(x)D_x(g(x)) = f'(x)g(x) + f(x)g'(x)$.

Rule 3. Quotient Rule. On an interval where $g(x) \neq 0$,

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_x\big(f(x)\big) - f(x)D_x\big(g(x)\big)}{\big(g(x)\big)^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2}$$

Rule 4. Power function derivative. If r is a real number, then

$$D_x(x^r) = rx^{r-1}$$

Examples: $D_x\left(x^{4/3}\right) = \frac{4}{3}x^{1/3}$, $\quad D_x\left(\frac{1}{x^{4/3}}\right) = -\frac{4}{3}\frac{1}{x^{7/3}}$, $\quad D_x\big(\sqrt{x}\,\big) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Rule 5. Chain Rule. On an open interval for which $(f\circ g)(x) \equiv f\big(g(x)\big)$ is defined

$$D_x\big((f\circ g)(x)\big) = f'(g(x))g'(x)$$

Example: $D_x\Big(\big(4+x^3\big)^5\Big) = (5)\big(4+x^3\big)^4(3x^2)$

Rule 6. Reciprocal Rule. On an interval where $f(x) \neq 0$ we have

$$D_x\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{\big(f(x)\big)^2}$$

Derivatives of trigonometric functions. First, we list the derivatives of the three basic functions.

$$D_x(\sin x) = \cos x \qquad D_x(\tan x) = \sec^2 x \qquad D_x(\sec x) = (\sec x)(\tan x)$$

Then we use this information to help determine the derivatives of the cofunctions.

Function	Derivative	Cofunction	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$

Note that the first column is simply the three basic functions. The second column is the derivative of the first column. The third column is a listing of the cofunctions of the first column. And to form the fourth column, begin by putting a minus sign in front of the expression to follow. Then, simply put the respective cofunctions of the second column after the minus sign.

Example: $D_x\big(\tan^3(\sqrt{x}\,)\big) = 3\tan^2(\sqrt{x}\,)\sec^2(\sqrt{x}\,)\dfrac{1}{2\sqrt{x}}$.

Derivative of exponential and logarithmic functions. $D_x\big(e^x\big) = e^x$ and $D_x(\ln|x|) = \dfrac{1}{x}$, $x \neq 0$. Also, for $a > 0$, $D_x\big(a^x\big) = (\ln a)a^x$.

We will limit our discussion on inverse trigonometric functions to

$$\sin^{-1} x \equiv \arcsin x \qquad \text{and} \qquad \tan^{-1} x \equiv \arctan x$$

We remind the reader that the exponents refer to inverse functions and **not** to reciprocals. The derivatives and corresponding anti-derivatives we will need are listed below:

$$D_x(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$D_x(\arctan x) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Now we turn our attention to anti-derivatives. The easiest place to start is with the power rule.

Anti-derivative of x^r. If $r \neq -1$, then $\displaystyle\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$.

The anti-derivatives of the basic trigonometric functions are found in the following table:

Function	Anti-derivative	Cofunction	Anti-derivative
$\sin x$	$-\cos x + C$	$\cos x$	$\sin x + C$
$\tan x$	$\ln\lvert\sec x\rvert + C$	$\cot x$	$\ln\lvert\sin x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$	$\csc x$	$\ln\lvert\csc x - \cot x\rvert + C$

There are other basic trigonometric anti-derivatives that are easily found by reversing the differentiation formulas.

Other trigonometric anti-derivatives.

$$\int \sec^2 x\,dx = \tan x + C \qquad \int (\sec x)(\tan x)\,dx = \sec x + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \int (\csc x)(\cot x)\,dx = -\csc x + C$$

Anti-derivatives of exponential functions. It is assumed that $a > 0$.

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{1}{\ln a}a^x + C$$

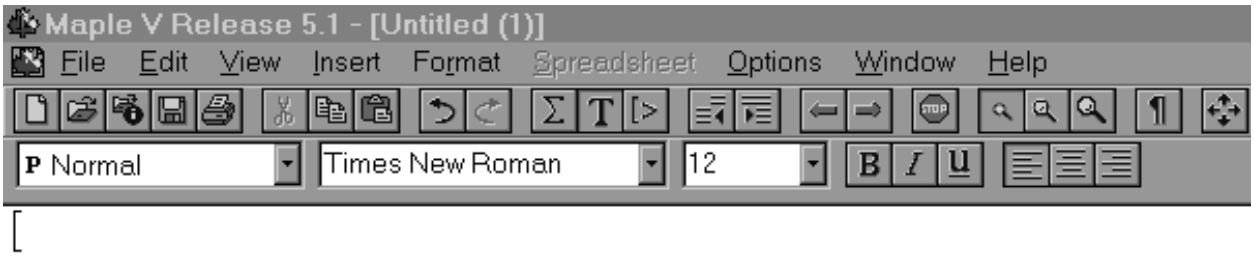
Review Exercises: Find derivatives of the following: 1. $x^2\cos x$ 2. $\dfrac{\ln x}{x^2+4}$ 3. $e^{3x}\tan(2x)$

4. Find an equation for the line tangent to the curve $y = \dfrac{x+3}{4+x^2}$ at the point where $x = 2$.

C2M0

Introduction to Maple V

Open Maple and obtain a blank worksheet. We are going to begin by establishing a format for each Maple assignment that is to be handed in. Do not type the "<" or ">" shown to identify your entries. And <Enter> means the "Enter" key. As you begin, the worksheet is in "math mode", so 'click' on the $\boxed{\mathbf{T}}$ to switch into "text mode".



Type <C2M1> <Enter> and then highlight C2M1 and click on the middle of the three boxes to the right of $\boxed{\mathbf{B}}$ $\boxed{I}$ $\boxed{u}$ so as to center C2M1. The left of these three buttons left-justifies text and the right one right-justifies it. Now,
<down arrow>, then type your name and section as shown.
<Midn Your Name> <Enter>
<Section> <Your section> <Enter>
Having completed this, highlight the three lines and then click on $\boxed{\mathbf{B}}$ to boldface everything. This is the format you should use for all Maple assignments to be handed in. For example, you should see something like

C2M1

Midn John Doe
Section 1234

You may eliminate the brackets on the left by the function key <F9>. To return to math mode, click on the $\boxed{[>}$. If we wanted to type a math formula while in text mode we would click on $\boxed{\Sigma}$. The class of 2001 has Release 4 of Maple V, and the classes of 2002 and 2003 have Release 5 or 5.1 . There are some differences between the releases and we will address them as we proceed. Later in this section we will discuss *palettes* which allow you to select commands from a menu and avoid using Maple syntax. It is the contention of the author of these notes that learning some Maple syntax is beneficial to the student, so even though you may accomplish the same things by clicking on a symbol, we will show you the syntax that would otherwise be hidden.

There are several packages of programs in Maple that we will find useful. For many calculus operations we will need "student". In math mode, lines in Maple end with a semicolon or colon. If you put a colon after the line then the display of the output is suppressed. Please type the command lines below in a new worksheet exactly as you see them and note the output. This work is for your benefit and is not intended to be handed in.

```
> with(student):
> A:=Int(x^2,x);
> value(A);
> B:=int(x^2,x);
> C=Int(x^2,x=2..5);
> value(C);
```

In the first line, the name "A" is being assigned to the unevaluated integral, $\int x^2\,dx$. Then we find the value of "A". By using the lowercase "i" we assign the name "B" to the value of the integral, rather than the inert expression that is not yet evaluated. By eliminating the colon, we have established an equation rather than assigning a name "C" to the integral. Think of it this way. The integral is being assigned to a piece of memory named "A", another piece of memory is assigned the value of that integral and is named "B", while an equation involving "C" and an unevaluated definite integral is established. Evaluating "C" is

meaningless because there is no piece of memory with that name. Go back, insert the colon as was done with "A" and hit <Enter> and then another <Enter>. The exact value of the integral, 39, should appear. Be very careful of these subtleties, because Maple cannot read your mind. What happens when you put a semicolon after '`with(student)`' instead of the colon?

In that same worksheet enter these command lines and observe the output.

```
> value(Pi^2/6);
> evalf(Pi^2/6);
> S100:=Sum(1/k^2,k=1..100);
> value(S100);
> evalf(S100);
```

The `evalf` command converts an exact numerical expression to a floating point number.

It is very important to know how to define a function, how to define an expression, and the difference between them. To define the function $f(x) = 4 - 2x - x^2$ please enter:

```
f:=x->4-2*x-x^2;
```

And then evaluate f at $x = 4/3$ by:

```
f(4/3);
```

Now define an expression *expr* by:

```
expr:=4-2*x-x^2;
```

In order to evaluate an expression at a value, we must substitute into the expression. Try:

```
subs(x=4/3,expr);
```

And when we try to use an expression as if it were a function, we get garbage. Try:

```
garbage:=expr(4/3);
```

It will be very useful later to be able to make a function out of an expression. The syntax for this is puzzling. Entering these commands should produce these results:

```
> P:=x^2+cos(x);
```

$$P := x^2 + \cos(x)$$

```
> G:=unapply(P,x);
```

$$G := x \to x^2 + \cos(x)$$

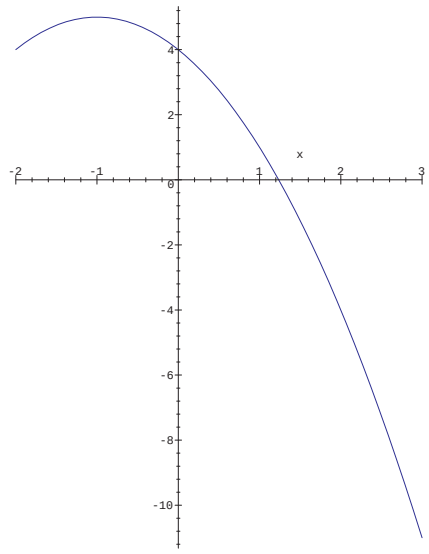
```
> G(Pi);
```

$$\pi^2 - 1$$

We see that G is a function and that $G(x) = P$.

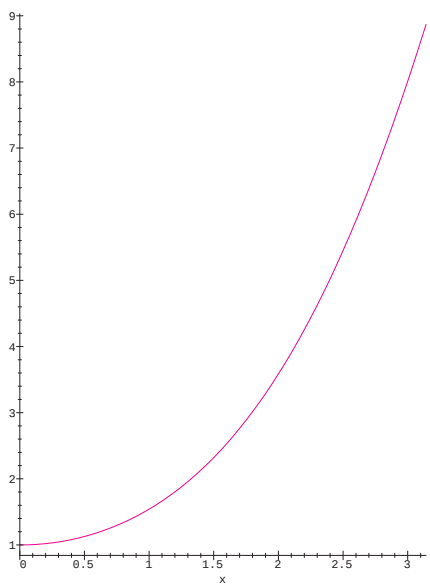
Maple graphics are versatile and easy to use. Above we defined $f(x) = x^2 + \cos(x)$ in our worksheet. We can get a quick plot of f on $[-2, 3]$ by:

```
> plot(f(x),x=-2..3,color=blue);
```



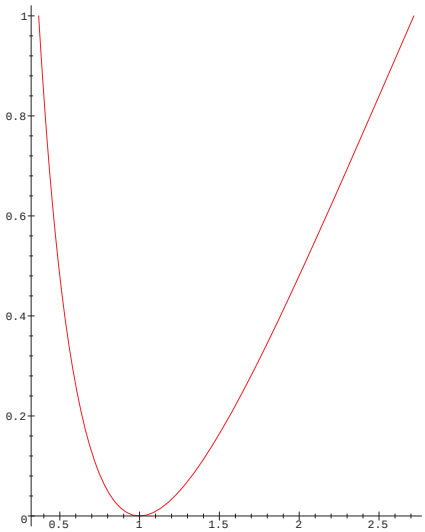
Note how we used $f(x)$ which is an expression, not just f, in this plot. To plot the function G from above we could use P or $G(x)$ and obtain identical results.

```
> plot(P,x=0..Pi,color=magenta);
```

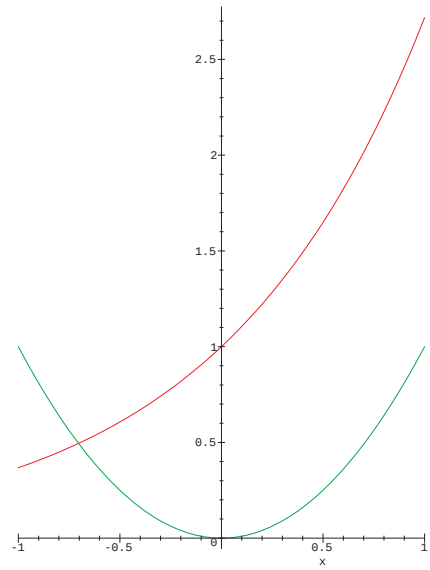


When you wish to plot two functions with the same domain it can be done very easily. However, it is very easy to confuse the syntax with parametric plotting. We will do an example of each so that you will know where to be careful. The placement of the righthand square bracket determines which format you have. In two-dimensional plotting, when you list two expressions and a range inside the square brackets the first function controls the value on the x-axis and the second function controls the value on the y-axis. This is parametric plotting. To save space the output follows on the left.

```
> plot([exp(x),x^2,x=-1..1]);
```



Parametric Plotting



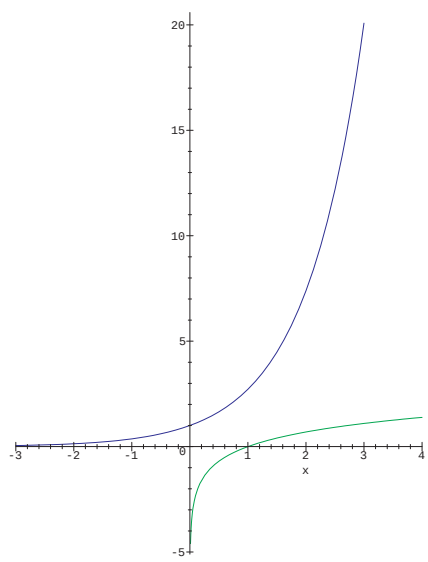
Two Functions

When you do not include the domain inside the square brackets you get two different plots on the same coordinate system as you can see above on the right. This was produced by:

```
> plot([exp(x),x^2],x=-1..1);
```

How should you plot two functions on the same coordinate axes which have different domains? The answer is to give each plot a name **using a colon at the end to suppress the output** and then display them together. The command `display` is in the graphics package called `plots`.

```
> with(plots):
> A:=plot(exp(x),x=-3..3):   ← colon!
> B:=plot(ln(x),x=.01..4):   ← colon!
> display(A,B);   ← semicolon!
```



There are two operations that are very basic in calculus, namely differentiation and integration, or anti-differentiation. Expressions in x and t can be differentiated with respect to either variable so we must remember to specify the variable. Using the expression $P = x^2 + \cos(x)$ from above we have

```
> Pprime:=diff(P,x);
```

$$Pprime := 2x - \sin(x)$$

And if we integrate P

```
> Pint:=int(P,x);
```

$$Pint := \frac{x^3}{3} + \sin(x)$$

Now let's do some of the same steps by using a palette. On your command line type

```
A:=
```

There are three palettes and to access them you begin by clicking on "View", then "Palettes". If you need symbols, select that palette, but for now we choose "expressions". You should see

Click on the box with the integral symbol $\boxed{\int a}$. Then click on the box with $\boxed{a^b}$. On your command line the cursor appears where you want x inserted, so you type $< x >$, and then **move to the next entry position by using the 'Tab' key**. Enter $< 2 >$, <Tab>, and then the variable of integration, $< x >$ and <Enter>. At the end of the command line put a semicolon and hit <Enter>. This should produce

```
> A:=int(x^2,x);
```

$$\frac{x^3}{3}$$

Suppose you have a question about some aspect of using Maple and the syntax is confusing you. On the command line simply type

```
<?topic>
```
<Enter>

and Maple will display what you need to learn about "topic". You can copy from the help page and paste on the command line as needed.

C2M1

Change of Variable

The art of changing the variable in an integration (anti-differentiation) problem must be practiced in order to master it. We learn this valuable tool by trial and error. Maple provides an easy environment in which to try different substitutions. One simple rule to remember when substituting in Maple is that $expr_{old} = expr_{new}$ is the order in which to write the change. Let's begin by doing a simple change of variable for $\int \cos^4(3x)\sin(3x)\,dx$ and then show how it can be done using Maple. Note that reference is made to the **third step** of the substitution. **Students who write out the steps of a substitution carefully, and are meticulous when applying the third step, make far fewer errors than their colleagues who do not. You have been warned!**

Example: $\displaystyle\int \cos^4(3x)\sin(3x)\,dx$ Let's try:

$$
\begin{aligned}
u &= \cos(3x) \\
du &= -3\sin(3x)\,dx \\
-\frac{1}{3}du &= \sin(3x)\,dx \qquad \text{(third step, } -\frac{1}{3}\,du \text{ replaces } \sin(3x)\,dx\text{)} \\
\int \cos^4(3x)\sin(3x)\,dx &= \int (u)^4(-\frac{1}{3}\,du) = -\frac{1}{3}\int u^4\,du \\
&= -\frac{1}{15}u^5 + C \qquad \text{now resubstitute, using } u = \cos(3x) \\
&= -\frac{1}{15}\cos^5(3x) + C
\end{aligned}
$$

Maple Example:

After we identify the integral by the name A and display it, we realize that $\cos(3x)$ will become "u". Since the "old" expression precedes the "new" one when doing a substitution in Maple, we write "$\cos(3x) = u$" when applying the Maple command `changevar`. Later when we resubstitute, we will use "$u = \cos(3x)$". Now consider the Maple worksheet below, paying attention to the order used in the substitutions:

```
> with(student):
> A:=Int((cos(3*x))^4*sin(3*x),x);
```

$$A := \int \cos(3x)^4 \sin(3x)\,dx$$

```
> B:=changevar(cos(3*x)=u,A);
```

$$B := \int -\frac{1}{3}u^4\,du$$

```
> B:=value(B);
```

$$B := -\frac{1}{15}u^5$$

```
> B:=subs(u=cos(3*x),B);
```

$$B := -\frac{1}{15}\cos(3x)^5$$

C2M1 Problems

These integral problems are to be done two ways. Do them with pencil and paper showing all details and do them with Maple using `changevar`.

1. $\displaystyle\int 24x\left(4+9x^2\right)^5\,dx$ 2. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ 3. $\displaystyle\int \frac{\left(9+\frac{5}{x}\right)^{3/2}}{x^2}\,dx$

4. $\displaystyle\int 12x\sin^{1/2}(3x^2)\cos(3x^2)\,dx$

C2M2

Rational Fractions or Partial Fraction Decompositions

When you were learning basic algebra there were probably problems assigned on adding fractions which had constants, linear expressions, and quadratic expressions in x in the numerators and denominators. Our objective here is to take the answers to those questions and find the fractions that were added together. The need here is to break down a complicated fraction into several simple ones whose anti-derivatives are (much!) easier to find. To approach this systematically we separate the problems into groups determined by the nature of the denominators of the fractions. Recall that $P(x) = x^2 - a^2$ can be factored into $P(x) = (x-a)(x+a)$, so we say that $P(x)$ is *reducible*. Likewise, $Q(x) = x^2 + a^2$ can **NOT** be factored, so $Q(x)$ is *irreducible*. There are different approaches to to these problems and we will use substitution as our method. Two principles from algebra are applicable here. The first states that if two polynomials in x agree for all values of x, then the polynomials have the same coefficients, that is, they are identical. The second states that $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

In the following, $P(x)$ is a polynomial in x and the degree of P is less than the degree of the denominator. For cases where the degree of P is greater than or equal to the degree of the denominator, one must

divide the polynomials and then work with the remainder. Note that we are forcing the left and right sides to agree for all x, so we may use $\equiv$ instead of $=$.

I. The denominator has distinct linear factors. The expression on the left breaks down as shown:

$$\frac{P(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} \equiv \frac{a_1}{x-r_1} + \frac{a_2}{x-r_2} + \cdots + \frac{a_n}{x-r_n}$$

and the problem is reduced to determining the coefficients $a_1, a_2, \ldots, a_n$. We are assuming that no two factors are the same.

Example: Decompose $\dfrac{7x-11}{(x-1)(x+1)(x-2)}$. Each factor occurs with a constant numerator.

$$\frac{7x-11}{(x-1)(x+1)(x-2)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{x-2} \qquad \text{multiply through by the LHS denominator}$$

$$7x-11 = a(x+1)(x-2) + b(x-1)(x-2) + c(x-1)(x+1) \qquad \text{substitute } x=1$$

$$-4 = a(2)(-1) \qquad \text{so } a=2 \qquad \text{substitute } x=-1$$

$$-18 = b(-2)(-3) \qquad \text{so } b=-3 \qquad \text{substitute } x=2$$

$$3 = c(1)(3) \qquad \text{so } c=1$$

From this we conclude:

$$\frac{7x-11}{(x-1)(x+1)(x-2)} = \frac{2}{x-1} - \frac{3}{x+1} + \frac{1}{x-2} \quad \text{for all values of } x \text{ except } 1,\ -1, \text{ and } 2.$$

II. The denominator has linear factors with repeat(s). In this case, each factor occurs as a denominator up to the power to which it occurs in the original denominator. It is easiest to explain with an example. Suppose the original denominator is $(x-1)^3(x+2)^2(x-4)$ then the decomposition will look like:

$$\frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{x+2} + \frac{e}{(x+2)^2} + \frac{f}{x-4}$$

Example: Decompose $\dfrac{3x^2-7x+1}{(x-1)^2(x+2)}$.

$$\frac{3x^2-7x+1}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2} \qquad \text{multiply by LHS denominator}$$

$$3x^2-7x+1 = a(x-1)(x+2) + b(x+2) + c(x-1)^2 \qquad \text{substitute } x=1$$

$$-3 = b(3) \qquad \text{so } b=-1, \text{ substitute that value and simplify}$$

$$3x^2-7x+1 = a(x-1)(x+2) - (x+2) + c(x-1)^2 \qquad \text{move } (x+2) \text{ to the other side}$$

$$3x^2-6x+3 = a(x-1)(x+2) + c(x-1)^2 \qquad \text{both sides must be divisible by } x-1, \text{ so divide}$$

$$3x-3 = a(x+2) + c(x-1) \qquad \text{use } x=1 \text{ again}$$

$$0 = 3a \qquad \text{so } a=0 \qquad \text{use } x=-2$$

$$-9 = c(-3) \qquad \text{so } c=3$$

And we conclude:

$$\frac{3x^2-7x+1}{(x-1)^2(x+2)} = \frac{-1}{(x-1)^2} - \frac{3}{x+2}$$

III. The denominator has an irreducible quadratic factor. A quadratic factor requires a linear numerator such as $ax+b$.

Example: Decompose $\dfrac{x^2+5}{(x-1)(x^2+2)}$.

$$\frac{x^2+5}{(x-1)(x^2+2)} = \frac{a}{x-1} + \frac{bx+c}{x^2+2} \qquad \text{multiply by LHS denominator}$$

$$x^2+5 = a(x^2+2) + (bx+c)(x-1) \qquad \text{substitute } x=1$$

$$6 = a(3) \qquad \text{so } a=2, \text{ substitute that value and simplify}$$

$$x^2+5 = 2x^2+4+(bx+c)(x-1)$$

$$-x^2+1 = (bx+c)(x-1) \qquad \text{both sides must be divisible by } x-1, \text{ so divide}$$

$$-x-1 = bx+c \qquad b=-1,\ c=-1$$

Our decomposition is:

$$\frac{x^2+5}{(x-1)(x^2+2)} = \frac{2}{x-1} - \frac{x+1}{x^2+2}$$

IV. The denominator has a repeated linear factor and an irreducible quadratic factor. The method illustrated here is the easiest one for this case. **This type of problem will be on the Final Exam**.

Example: Decompose $\dfrac{8x}{(x-1)^2(x^2+3)}$.

$$\frac{8x}{(x-1)^2(x^2+3)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+3} \qquad \text{multiply by LHS denominator}$$

$$8x = a(x-1)\left(x^2+3\right) + b\left(x^2+3\right) + (cx+d)(x-1)^2 \qquad \text{substitute } x=1$$

$$8 = 4b \qquad \text{so } b=2, \text{ substitute that value}$$

$$8x = a(x-1)\left(x^2+3\right) + 2x^2+6 + (cx+d)(x-1)^2 \qquad \text{move } 2x^2+6 \text{ to the LHS}$$

$$-2x^2+8x-6 = a(x-1)\left(x^2+3\right) + (cx+d)(x-1)^2 \qquad x-1 \text{ is a factor of RHS and LHS, } \div$$

$$-2(x-3) = a\left(x^2+3\right) + (cx+d)(x-1) \qquad \text{substitute } x=1$$

$$4 = 4a \qquad \text{so } a=1, \text{ substitute that value}$$

$$-2x+6 = x^2+3+(cx+d)(x-1) \qquad \text{move } x^2+3 \text{ to the LHS}$$

$$-x^2-2x+3 = (cx+d)(x-1) \qquad \text{divide by } x-1$$

$$-x-3 = cx+d \qquad \text{which forces } c=-1 \text{ and } d=-3$$

$$\frac{8x}{(x-1)^2(x^2+3)} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{x+3}{x^2+3}$$

Now observe how Maple would have done the problem.

Maple Example:

```
> with(student):
> convert(8*x/((x-1)^2*(x^2+3)),parfrac,x);
```

$$\frac{2}{(x-1)^2} + \frac{1}{x-1} - \frac{3+x}{x^2+3}$$

C2M2 Problems

Find the partial fraction decompositions of the expressions using pencil and paper. Then, check your answers using Maple as was done in the preceding Maple Example. Do **NOT** use Maple to duplicate your pencil and paper work.

1. $\dfrac{8x+5}{(x+1)^2(x^2+2)}$ 2. $\dfrac{4x^2+4x+12}{x^2(x^2+4)}$ 3. $\dfrac{7x^2-17x+1}{(x-2)^2(x^2+1)}$ 4. $\dfrac{x^3-2x^2+2x-1}{x^3(x^2+1)}$

C2M3

Simpson's and Trapezoidal Rule

Riemann sums, Simpson's Rule, and the Trapezoidal Rule are available in Maple in the Student package. The example chosen here involves the sine function on the interval $[1,3]$ using 60 subintervals for the sums and 10 subintervals for the graphics. Because we wanted the decimal or floating point answer we used `evalf` instead of `value`, which would have listed a long summation. The actual integral is $\int_1^3 \sin x\,dx = -\cos 3 + \cos 1 \approx 1.530294803$. Please observe the output of each command line below.

```
> restart:    with(student):
> simpson(sin(x),x=1..3,60);
```

$$\frac{1}{90}\sin(1) + \frac{1}{90}\sin(3) + \frac{2}{45}\left(\sum_{i=1}^{30}\sin\left(\frac{29}{30}+\frac{1}{15}i\right)\right) + \frac{1}{45}\left(\sum_{i=1}^{29}\sin\left(1+\frac{1}{15}i\right)\right)$$

```
> evalf(%);
```

$$1.530294813$$

```
> trapezoid(sin(x),x=1..3,60);
```

$$\frac{1}{60}\sin(1) + \frac{1}{30}\left(\sum_{i=1}^{59}\sin\left(1+\frac{1}{30}i\right)\right) + \frac{1}{60}\sin(3)$$

```
> evalf(%);
```

$$1.530153105$$

```
> Int(sin(x),x=1..3);
```

$$\int_1^3 \sin(x)\,dx$$

```
> value(%);
```

$$-\cos(3) + \cos(1)$$

```
> evalf(%);
```

$$1.530294803$$

Accuracy The error estimates for the Trapezoidal Rule and Simpson's Rule are stated in the course textbook. As a reminder, if $|f''(x)| \le K$ and $|f^{(iv)}(x)| \le M$, and n subintervals are used, then the errors for the respective rules, E_T and E_S, satisfy

$$|E_T| \le \frac{K(b-a)^3}{12\,n^2} \qquad \text{and} \qquad |E_S| \le \frac{M(b-a)^5}{180\,n^4}$$

when applied over the interval $[a,b]$.

Trapezoidal Rule Maple Example Approximate $\displaystyle\int_0^{\pi/3} \sin(2x)\,dx$ to within $\frac{1}{1000}$ using the Trapezoidal Rule. Determine the number of subintervals necessary to achieve the requested accuracy by applying the estimate displayed above. It is best if our second derivative is a function, so we use `unapply` to create a function from an expression.

```
> restart:    with(student):
> f:=x->sin(2*x);
```

$$f := x \to \sin(2x)$$

```
> f2:=unapply(diff(f(x),x,x),x);
```

$$f2 := x \to -4\sin(2x)$$

At this point we have the second derivative of f as a function. We must find the maximum value of the absolute value of the second derivative on the interval.

```
> K:=maximize(abs(f2(x)),x,{x=0..evalf(Pi/3)});
```

$$K := 4$$

Equate the error and the overestimate and solve for the value of n that works.

```
> Eqn1:=(Pi/3-0)^3*K/(12*n^2)=1/1000;
```

$$Eqn1 := \frac{1}{81}\frac{\pi^3}{n^2} = \frac{1}{1000}$$

```
> solve(Eqn1,n);
```

$$\frac{10}{9}\sqrt{10}\,\pi^{(3/2)}, -\frac{10}{9}\sqrt{10}\,\pi^{(3/2)}$$

```
> evalf(%);
```

$$19.56511025, -19.56511025$$

Since n must be an integer, choose $n = 20$

```
> app:=trapezoid(f(x),x=0..Pi/3,20);
```

$$app := \frac{1}{120}\pi\left(2\left(\sum_{i=1}^{19}\sin\left(\frac{1}{30}i\pi\right)\right)+\frac{1}{2}\sqrt{3}\right)$$

```
> approx:=evalf(app);
```

$$approx := .7493144853$$

```
> ans:=evalf(int(f(x),x=0..Pi/3));
```

$$ans := .7500000000$$

```
> abs(approx-ans);
```

$$.0006855147$$

So, we have achieved the requested accuracy.

C2M3 Problems:

1. Use Maple to find the requested approximations. $\int_0^2 \sqrt{1+x^2}\,dx$, $n = 40$, use `simpson, trapezoid`

2. Modify the Maple Example above and use Simpson's Rule instead of the Trapezoidal Rule to approximate $\int_0^2 \frac{x^2}{1+x^4}\,dx$ to within $\frac{1}{1000}$.

3. Modify the Maple Example above and then use Simpson's Rule instead of the Trapezoidal Rule to approximate $\int_0^1 x\arctan(x)\,dx$ to within $\frac{1}{10000}$.

C2M4

Improper Integrals

Improper integrals can occur in two different ways. The interval of integration can be unbounded, or the integrand can be an unbounded function. As you know, in the respective cases

$$\int_a^\infty f(x)\,dx \equiv \lim_{M\to\infty}\int_a^M f(x)\,dx$$

$$\int_a^b f(x)\,dx \equiv \lim_{t\to a^+}\int_t^b f(x)\,dx$$

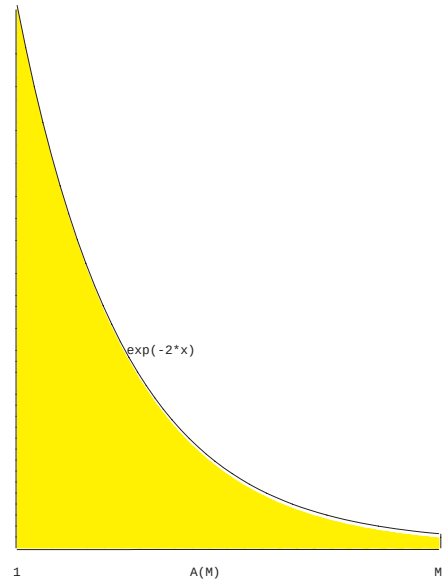
where, in the second case f is unbounded at a. In Maple, we will evaluate the integral from a to M and then evaluate the limit of that result as the definition suggests in order to reinforce the concepts.

Maple Example:

Evaluate the improper integral $\int_1^{\infty} e^{-2x}\,dx$

We find an expression $A(M)$ which expresses the area of the region under the curve as a function of the righthand endpoint, M. Then we evaluate the limit

$$\lim_{M\to\infty} A(M)$$




```
> with(student):
> A:=Int(exp(-2*x),x=1..M);
```

$$A := \int_1^M e^{(-2x)}\,dx$$

```
> A:=value(A);
```

$$A := -\frac{1}{2}e^{(-2M)} + \frac{1}{2}e^{(-2)}$$

```
> limit(A,M=infinity);
```

$$\frac{1}{2}e^{(-2)}$$

Maple Example: Evaluate the improper integral $\int_0^1 \frac{1}{x^{1/3}}\,dx$ Note that the integrand is discontinuous at the lefthand endpoint.

```
> B:=Int(x^(-1/3),x=t..1);
```

$$B := \int_t^1 \frac{1}{x^{1/3}}\,dx$$

```
> B:=value(B);
```

$$B := \frac{3}{2} - \frac{3}{2}t^{2/3}$$

```
> limit(B,t=0);
```

$$\frac{3}{2}$$

C2M4 Problems Evaluate the improper integrals using Maple.

1. $\int_2^{\infty} \frac{1}{x^{4/3}}\,dx$
2. $\int_3^{\infty} \frac{\ln x}{x}\,dx$
3. $\int_0^8 x^{-2/3}\,dx$
4. $\int_0^1 x\ln x\,dx$
5. $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

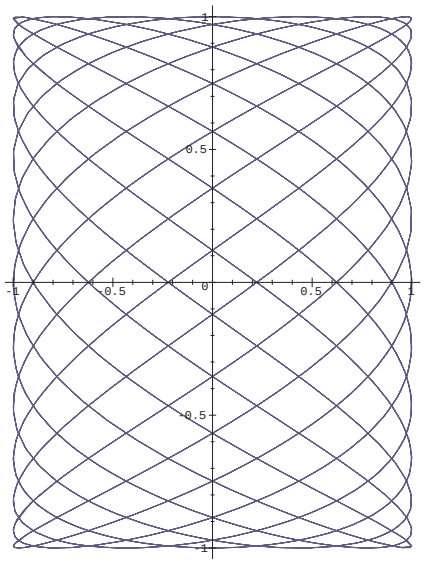
C2M5

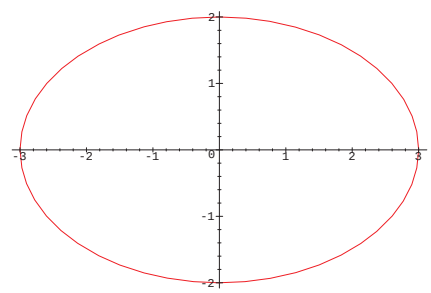
Parametric Functions

Have you ever played with a toy called *"Etch-a-Sketch"*? One hand controls the x-axis while the other controls the y-axis. It is as if you are graphing $(x(t), y(t))\,, a \le t \le b\,,$ which is exactly what happens when a function in the plane is defined parametrically. Be very careful where you place the right bracket,], when using Maple to plot parametric graphs.

Maple Example: Plot $x(t) = \sin(13t)\,,\ y(t) = \cos(7t)$ for $0 \le t \le 6\pi$ which produces a *lissajou*. The plot is on the left below. As you can see, the scaling is a little off because the "square" is two units on each side. For a little fun, increase the coefficients to say 43 and 37 and see what happens. You may also wish to increase the domain.

```
> plot([sin(13*t),cos(7*t),t=0..6*Pi],color=navy);
```





Maple Example: Ellipses are easy this way. Plot $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$. The Maple output is above on the right. When you have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ you may plot this by using $x(t) = a\,\cos(t)$ and $y(t) = b\,\sin(t)$ for $0 \le t \le 2\pi$. So,

```
> plot([3*cos(t),2*sin(t),t=0..2*Pi]);
```

C2M5 Problems Use Maple to display the parametric graphs of the given functions.

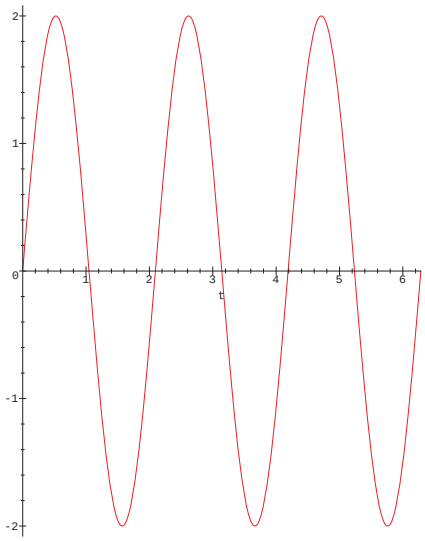
1. $x = e^t,\ y = e^{2t},\ -1 \le t \le 2$
2. $x = 2\sec t,\ y = \tan t,\ -\pi/2 < t < \pi/2$
3. $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 4\pi$
4. $x = \cos^3 t,\ y = \sin^3 t,\ 0 \le t \le 2\pi$

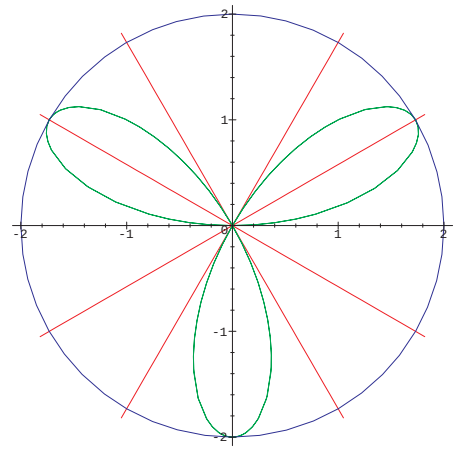
C2M6

Polar Coordinates

For many graphs in polar coordinates the easiest place to start is with a rectangular plot. We will call these *helper diagrams*. For the function $r = 2\sin(3\theta)$ we have the rectangular plot on the left and the polar plot on the right. We can see from the helper diagram that as θ varies from 0 to $\pi/3$ the radius will start at 0, reach 1 at $\pi/6$, and decrease back to 0 at $\pi/3$. The next thing we observe is that from $\pi/3$ to $2\pi/3$ the radius will take on negative values. At each point where the helper diagram attains the value 2, the polar graph will be two units from the origin for that angle. And, whenever the helper diagram has the value of -2, the polar graph will be two units in the opposite direction of that angle. At points on the helper diagram where the value is 0, the polar graph will be tangent to that angle as the graph passes through the polar origin. To sketch the polar plot, begin by drawing a circle of radius 2, then draw radial lines for

multiples of $\pi/6$. Mark the intersection for each peak and valley of the helper diagram. These have polar coordinates whose angles correspond to odd multiples of $\pi/6$. Remember, sometimes it is a 2 and sometimes it is a radius of -2. On the polar graph, put your finger at the origin and trace the plot, realizing that you are going to go around the plot twice.

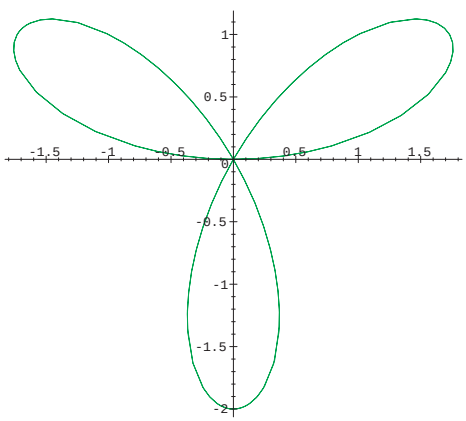


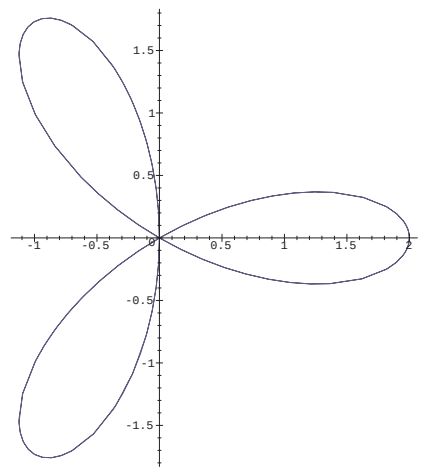


Plotting in polar coordinates is very easy using Maple. We need the library `plots` in order to use `polarplot`. We will use t instead of θ as the variable sometimes.

Maple Example 1: Plot $r = 2\sin(3\theta)$. On the right, $r = 2\cos(3\theta)$ is displayed so that you may compare.

```
> with(plots):
> polarplot(2*sin(3*t),t=0..2*Pi,color=green);
```

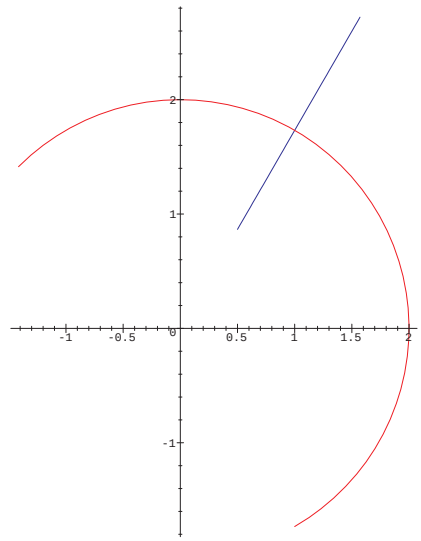




Maple Example 2: Polarplot may also be used parametrically. We will define two plots and give them names, *A1* and *B1*, and end their lines with colons to suppress the output. Using `polarplot` parametrically, the first coordinate determines r and the second determines θ. Each may be functions of a third variable whose domain must be specified within the square brackets.

Maple Example 3:

```
> with(plots):
> A1:=polarplot([r,Pi/3,r=1..Pi],color=blue):
> B1:=polarplot([2,theta,theta=-Pi/3..3*Pi/4]):
> display({A1,B1});
```

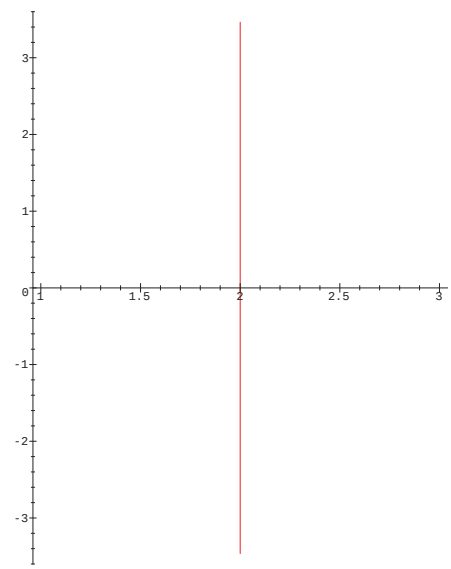


Maple Example 4: Suppose we consider the vertical line $x = 2$. We know that $x = r\cos(\theta)$ in polar coordinates, so set the x values equal. This means that $r\cos(\theta) = 2$ and that we may solve for r, which produces

$$r = \frac{2}{\cos(\theta)} = 2\sec(\theta)$$

We must avoid dividing by 0, so odd multiples of $\pi/2$ must be avoided. Here we will stay between $-\pi/3$ and $\pi/3$.

```
> with(plots):
> polarplot([2*sec(t),t,t=-Pi/3..Pi/3]);
```



C2M6 Problems Use Maple to determine the polar graphs of the given functions. Remember, you must set the domain. Also, resize the output to a reasonable size. Save paper.

1. $r = 1 + \cos(\theta)$ 2. $r = 2\csc(\theta)$ 3. $r = \sqrt{2} - 2\cos(\theta)$ 4. $r = \theta,\ -\pi \le \theta \le 2\pi$

C2M7

Solutions of Differential Equations

A differential equation arises when there is a relationship involving a function and one or more of its derivatives. For example

$$y'' + 5y' + 6y = 0$$

is such an equation. A function is a solution of this equation if you obtain 0 when you add its second derivative to 5 times its first derivative and then add 6 times the function itself.

Maple Example 1 Use Maple to verify that $y(t) = ae^{-3t} + be^{-2t}$ is a solution of the differential equation shown above, where a and b are arbitrary constants.

```
> with(student):
> de1:={diff(y(t),t,t)+5*diff(y(t),t)+6*y(t)=0};
```

$$de1 := \{\left(\frac{\partial^2}{\partial t^2} y(t)\right) + 5\left(\frac{\partial}{\partial t} y(t)\right) + 6\, y(t) = 0\}$$

```
> y1:=a*exp(-3*t)+b*exp(-2*t);
```

$$y1 := ae^{(-3t)} + be^{(-2t)}$$

```
> eval(de1,y(t)=y1);
```

$$\{0 = 0\}$$

which shows that for any constants a and b, $y(t)$ is a solution of the given equation.

Maple Example 2 Determine whether $y(x) = e^x + ce^{-2x}$ is a solution of

$$y' + 2y = 3e^x$$

for any value of the constant c.

```
> de2:={diff(y(x),x)+2*y(x)=3*exp(x)};
```

$$de2 : \{\left(\frac{\partial}{\partial x} y(x)\right) + 2y(x) = 3e^x\}$$

```
> y2:=exp(x)+c*exp(-2*x);
```

$$y2 := e^x + ce^{(-2x)}$$

```
> eval(de2,y(x)=y2);
```

$$\{3e^x = 3e^x\}$$

How would we know if we did not have a solution? let's define a different function and see what happens.

```
> y3:=2*exp(x)+C*exp(-2*x);
```

$$y3 := 2e^x + Ce^{(-2x)}$$

```
> eval(de2,y(x)=y3);
```

$$\{6e^x = 3e^x\}$$

Now in order for $y3$ to be a solution, the last equation, $6e^x = 3e^x$, would have to be true for every x. But this is true for *no* x, so $y3$ is not a solution.

C2M7 Problems: Use Maple and the method illustrated above to determine whether the given function is a solution of the differential equation.

1. $y = \sin x + x^2$, $\quad y'' + y = x^2 + 2$
2. $y = e^{2x} - 3e^{-x}$, $\quad y'' - y' - 2y = 0$
3. $x = 2e^{3t} - e^{2t}$, $\quad \dfrac{d^2x}{dt^2} - x\dfrac{dx}{dt} + 3x = -2e^{2t}$
4. $x = \cos 2t$, $\quad \dfrac{dx}{dt} + tx = \sin 2t$
5. $x = \cos t - 2\sin t$, $\quad x'' + x = 0$

Maple V Spreadsheets

In Release 5 or 5.1 of Maple V the user may insert a spreadsheet into the Maple worksheet. One characteristic of these spreadsheets that must be noted early is that changing an entry does not cause the spreadsheet to immediately recalculate as is the case in the standard spreadsheet. This is easily done by clicking on a button, however. The reader is reminded that the easiest way to get help or information about spreadsheets in Maple is to enter a worksheet and type in

```
> ?spreadsheet <enter>
```

and a list of topics is displayed. The reader is warned that there is an error in the identification of row and column headers in the section that tries to explain which is which. Column headers are listed in a row as A, B, C, etc. while row headers are listed next to the first column as 1, 2, 3, etc. This is reversed in the diagram you would see in the help display.

Please open a blank worksheet and enter the following lines:

```
> with(student):
> f:=x->2+2*x-x^2;
```

$$f := x \to 2 + 2x - x^2$$

```
> g:=x->sin(2*x);
```

$$g := x \to \sin(2x)$$

With the mouse, move the cursor arrow to the command **Insert**, click and then click on **Spreadsheet**. In addition to the material already entered, you should see a blank spreadsheet ready to be resized to fit your needs. Before we do that, note that the menu bars above have changed and that they look like this:

Before you inserted the spreadsheet, you could not select **Spreadsheet**, but now it is an option. Click on it and check out the menu. Note the four boxes on the left and below the main bar. The one to the left is very useful. When you need to fill cells in a spreadsheet in some direction, click on this box. More specific instructions follow when this process is needed in our example.

Now we are ready to resize the blank spreadsheet. Outside the spreadsheet, but near the lower righthand corner, click the mouse. A drag box outline should appear around the spreadsheet. Then, move the cursor to that corner until the diagonal arrow appears. Move that corner down and to the left so that you end up with five columns and 15 or so rows. This might take several tries. When you have done this, click on the cell in row 1 and column A. In the first row, enter *x*, *f(x)*, *x*, and *g(x)*. The arrow keys will move from one cell to another. You will note that the value of the function has appeared where you entered $f(x)$ and $g(x)$. We are going to illustrate different methods to accomplish the same things. In cell **A2** enter 0 and move to cell **B2**. One option here is to enter `f(~A2)` and we will do that. The other is to enter `eval(~$B$1,x=~A2)`. The dollar signs mean that cell **B1** will always be used and not just refer to the cell directly above. This is an *absolute* reference rather than a *relative* one.

Our objective now is to fill in the two columns, **A** and **B**. Click on **A2** and highlight down to cell **A12**. The menus at the top of the screen change when you are in a spreadsheet. At the extreme left and on the third row of the menus, you will find a button that looks like three window panes and the shade is pulled down in the top one. It also has an arrow pointing down. Click on this button and a menu pops up. Click on the window that indicates 'step size' and enter .1 , then click on **OK**. You should see the first column of the completed spreadsheet. Move to cell **B2** and highlight down to **B12**. On the top line of the menus click **Spreadsheet**, click on **Fill - Down**. You don't actually click on **Fill** because when the arrow touches it, the side menu with **Down** pops up immediately. This should complete the second column.

Before continuing with the spreadsheet entries, move the cursor outside of the spreadsheet to the command line above the spreadsheet and hit <Enter>. This should cause a command line to appear below the spreadsheet. Enter $M :=$ on that line and then highlight the cells in the first two columns from **A2** to **B12**.

Click on the **Copy** button, move the cursor to after the $M :=$ on the line below, and click on the **Paste** button. Immediately put a colon at the end of the line and then <Enter>. The colon suppresses the output, which is a matrix of 2×2 matrices. On the next line enter `c:=convert(M,listlist);` and a list of those 2×2 matrices is shown. We can use this list in an interesting way. Enter `plot(c,style=line);` and <Enter>, and a plot appears. It is rather smooth because our x values are close together.

Let's return to the spreadsheet and put 0 in cell **C2**. Move down to cell **C3** and enter `~C2+Pi/12` . Then, highlight from **C2** to **C12** and click on **Spreadsheet, Fill - Down**. In cell **D2** enter `eval(~$D$1,~C2)` and 0 appears. Highlight **D3** down to **D12**, then click on **Spreadsheet, Fill - Down**. This completes the spreadsheet.

Here is the tricky part. To plot the second function, highlight the rectangular area **C2** to **D12**, copy and paste to the command line below the spreadsheet, insert
`M1:=` before `MATRIX`,
a colon `:` at the end of the last line, and <Enter>.
On the next line put
`c1:=convert(M1,listlist);` <Enter>
and follow that line with
`plot(c1,style=line);` and <Enter>,
and a polygonal graph appears. The spreadsheet is on the right and the remainder of the worksheet follows.

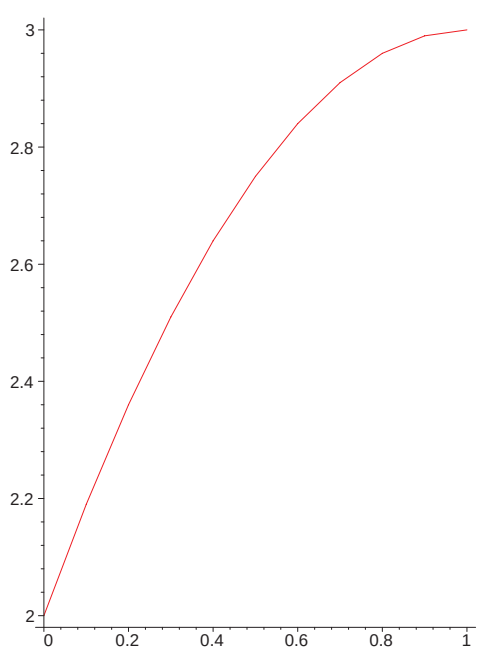
	A	B	C	D
1	x	$2+2x-x^2$	x	$\sin(2x)$
2	0	2	0	0
3	.1000	2.1900	$\frac{1}{12}\pi$	$\frac{1}{2}$
4	.2000	2.3600	$\frac{1}{6}\pi$	$\frac{1}{2}\sqrt{3}$
5	.3000	2.5100	$\frac{1}{4}\pi$	1
6	.4000	2.6400	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$
7	.5000	2.7500	$\frac{5}{12}\pi$	$\frac{1}{2}$
8	.6000	2.8400	$\frac{1}{2}\pi$	0
9	.7000	2.9100	$\frac{7}{12}\pi$	$\frac{-1}{2}$
10	.8000	2.9600	$\frac{2}{3}\pi$	$-\frac{1}{2}\sqrt{3}$
11	.9000	2.9900	$\frac{3}{4}\pi$	-1
12	1	3	$\frac{5}{6}\pi$	$-\frac{1}{2}\sqrt{3}$

```
> M:=MATRIX([[0, 2], [.1, 2.19], [.2, 2.36], [.3, 2.51], [.4, 2.64], [.5, 2.75],
    [.6, 2.84], [.7, 2.91],[.8, 2.96], [.9, 2.99], [1, 3]]):
```

```
> c:=convert(M,listlist);
```

c := [[0, 2], [.1, 2.19], [.2, 2.36], [.3, 2.51], [.4, 2.64], [.5, 2.75], [.6, 2.84], [.7, 2.91], [.8, 2.96], [.9, 2.99], [1, 3]]

```
> plot(c,style=line);
```

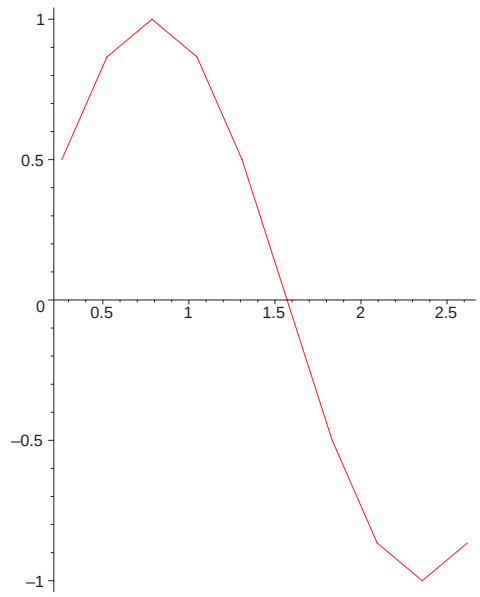


```
> M1:=MATRIX([[1/12*Pi, 1/2], [1/6*Pi, 1/2*sqrt(3)], [1/4*Pi, 1], [1/3*Pi, 1/2*sqrt(3)],
 [5/12*Pi, 1/2],[1/2*Pi, 0], [7/12*Pi, -1/2], [2/3*Pi, -1/2*sqrt(3)], [3/4*Pi, -1],
 [5/6*Pi, -1/2*sqrt(3)]]):
```

```
> c1:=convert(M1,listlist);
```

$$c1 := [[\frac{1}{12}\pi, \frac{1}{2}], [\frac{1}{6}\pi, \frac{1}{2}\sqrt{3}], [\frac{1}{4}\pi, 1], [\frac{1}{3}\pi, \frac{1}{2}\sqrt{3}], [\frac{5}{12}\pi, \frac{1}{2}], [\frac{1}{2}\pi, 0], [\frac{7}{12}\pi, \frac{-1}{2}],$$
$$[\frac{2}{3}\pi, \frac{-1}{2}\sqrt{3}], [\frac{3}{4}\pi, -1], [\frac{5}{6}\pi, \frac{-1}{2}\sqrt{3}]]$$

```
> plot(c1,style=line);
```



Do you remember the array of trigonometric derivatives in the Review section at the beginning of these notes? It should be easy to duplicate using a Maple spreadsheet. Open a spreadsheet and in the first row insert *function*, *derivative*, *cofunction*, *derivative*. In the first column, **A**, starting with **A2**, enter *sin(x)*, *tan(x)*, and *sec(x)*. Move to **B2** and enter `diff(~A2,x)`. Highlight **B2** to **B4**, then click on **Spreadsheet, Fill - Down** successively. Enter the cofunctions in Column **C**, and the derivatives in column **D** similar to how you handled column **B**.

	A	B	C	D
1	*function*	*derivative*	*cofunction*	*derivative*
2	$\sin(x)$	$\cos(x)$	$\cos(x)$	$-\sin(x)$
3	$\tan(x)$	$1+\tan(x)^2$	$\cot(x)$	$-1-\cot(x)^2$
4	$\sec(x)$	$\sec(x)\tan(x)$	$\csc(x)$	$-\csc(x)\cot(x)$

Let's consider one more example. Start a worksheet, define a function $f(x) = \cos(2x)$, and set $a = Pi/6$ Open a spreadsheet and in cell **A1** put $f(x)$. Move to **A2** and enter `diff(~A1,x)`. Highlight **A2** down to **A8**. Click on **Spreadsheet, Fill - Down** and note that succesive derivatives appear. In **B1**, put `eval(~A1,x=a)` and then highlight **B1** down to **B8**. Click on **Spreadsheet, Fill - Down**. When you reach the section on Taylor series you will realize just how useful a spreadsheet like this example can be.

C2M8

Direction Fields and Euler's Method

Direction Fields As we begin our study of differential equations we come across equations of the form $y' = f(x, y)$, which means that at each point in some region the slopes of solutions of the equation are known. It is very useful to be able to plot short line segments at a reasonable number of points and thereby have a picture of the behavior of these slopes. The command we need in Maple is found in the package 'plots' and the problem is that the input must have two coordinates, not one. We can get around this by making the first coordinate equal to 1 and the second the value of $f(x, y)$.

Maple Example: Suppose we are given a direction field $y' = x + y$ and we wish to plot it.

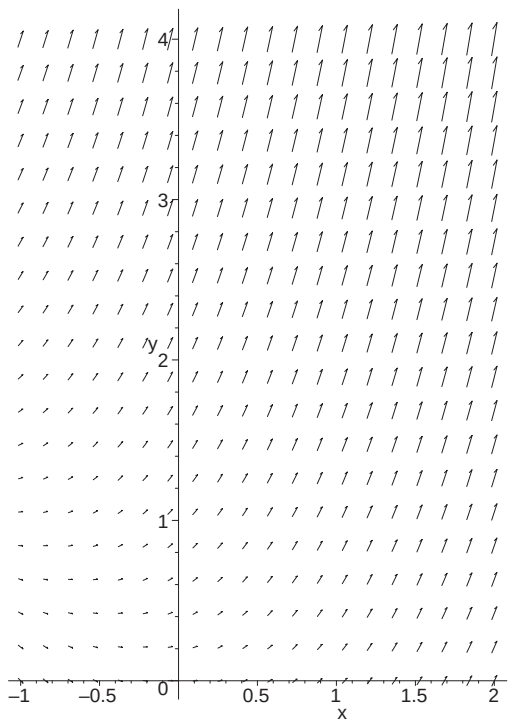
```
> with(plots):
> f:=(x,y)->x+y;
```

$$f := (x, y) \to x + y$$

```
> F:=(x,y)->[1,f(x,y)];
```

$$F := (x, y) \to [1, f(x, y)]$$

```
> fieldplot(F(x,y),x=-1..2,y=0..4);
```



Euler's Method This method assumes that a direction field, $y' = f(x, y)$, and a boundary value, $y_0 = y(x_0)$, are known for a potential solution y. Starting at x_0, a linear approximation is used to locate a second point and this process is repeated until an approximate solution for the equation is found for $x = x_n$. Let's begin by recalling the *point-slope* form for the equation of a line:

$$y - y_0 = m(x - x_0)$$

Identify $h = x - x_0$ and $m = f(x_0, y_0)$ and you have

$$y = y_0 + f(x_0, y_0)h \quad \text{or}$$
$$y_{i+1} = y_i + f(x_i, y_i)h$$

when this is viewed as an iterative process. As you will soon see, spreadsheets are quite useful here.

Maple Example Let's consider the direction field from our example above and the boundary value $y(0) = 1$ and approximate the value of $y(1)$ by using a spreadsheet. For simplicity, we will redefine f in this worksheet. The readers are urged to do the steps on their computers as they read along.

```
with(plots):
> f:=(x,y)->x+y;
```

$$f := (x, y) \to x + y$$

Click on **Insert** and **Spreadsheet** and resize the spreadsheet so that there are 3 columns and 12 rows. In the first row insert *n, xn, yn*, move to the second row and insert 0, 0, 1. That last value is $y(0)$.
First column. Highlight the first column from **A2** to **A12**, and then click on the button on the extreme left of the third row of the menus at the top of the page. The button looks like a window with three vertical panes and the blind pulled down over the top pane, along with an arrow pointing down. In the menu box that appears, insert a step size of 1 and a stop value of 10.

Second column. Highlight the second column from **B2** to **B12**, click on the button above on the extreme left (same as before), and insert a step size of .1 and a stop value of 1.

Third column. Carefully insert into **C3**: `~C2+(.1)*f(~B2,~C2)` and <Enter>. Note how this entry will reflect the old value of y plus the slope times the change in x, as it should. To complete this column, highlight from **C3** to **C12**, click on **Spreadsheet**, **Fill**, and **Down**. The spreadsheet should look like the one displayed below.

	A	B	C
1	*n*	*xn*	*yn*
2	0	0	1
3	1	.1000	1.1000
4	2	.2000	1.2200
5	3	.3000	1.3620
6	4	.4000	1.5282
7	5	.5000	1.7210
8	6	.6000	1.9431
9	7	.7000	2.1974
10	8	.8000	2.4871
11	9	.9000	2.8158
12	10	1	3.1874

There should be a prompt > below the spreadsheet. Enter `M :=` and then highlight the region in the spreadsheet from **B2** to **C12**, click on the **Copy** button at the top, move the cursor to after the `M:=` on the line below and then click on the **Paste** button at the top. Insert a colon at the end of that line, and then <Enter>. You should see:

```
> M:=MATRIX([[0, 1], [.1, 1.1], [.2, 1.22], [.3, 1.362], [.4,
1.5282], [.5, 1.72102], [.6, 1.943122], [.7, 2.1974342], [.8,
2.48717762], [.9, 2.815895382], [1, 3.187484920]]):
>
```

Now we must pair the points together by converting this *matrix* of 1×2 matrices to a *list* of 1×2 matrices. Then we will be able to treat each 1×2 matrix as a point in the plane and plot the graph.

```
> a:=convert(M,listlist);
```

$$a := [[0, 1], [.1, 1.1], [.2, 1.22], [.3, 1.362], [.4, 1.5282], [.5, 1.72102], [.6, 1.943122], [.7, 2.1974342], [.8, 2.48717762], [.9, 2.815895382], [1, 3.187484920]]$$

We are going to plot these points and join them together with line segments, and then display the plot at the same time that we display the direction field. This means that we must give these plots names and suppress their output until we are ready.

```
> A:=pointplot(a,style=line,color=blue):
```

Now to construct a function that Maple can interpret as acceptable for the command `fieldplot`:

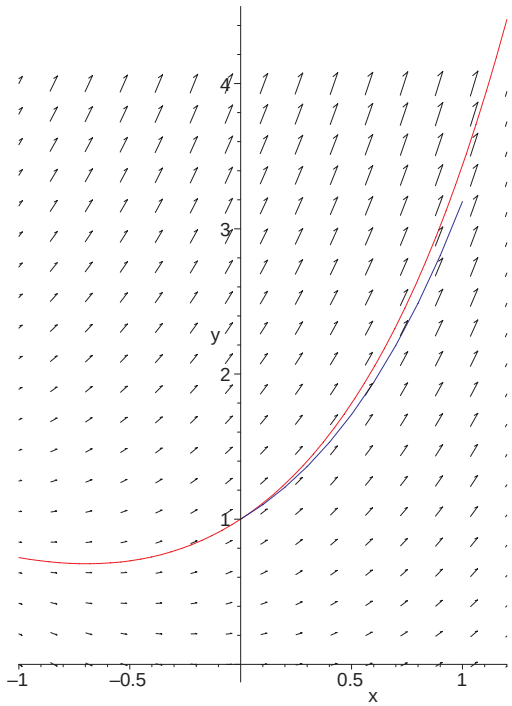
```
> F:=(x,y)->[1,f(x,y)];
```

$$F := (x, y) \to [1, f(x, y)]$$

```
> B:=fieldplot(F(x,y),x=-1..2,y=0..4):
```

Because we happen to know the solution to this equation we will include it in the plot.

```
> C:=plot(2*exp(x)-x-1,x=-1..1.2,color=red):
> display(A,B,C);
```



C2M8 Problem Use Maple to graph an approximation to the solution of the differential equation

$$y' = x - y^2 \qquad y(1) = -1$$

Use a step size of 0.1 and approximate $y(3)$. This requires twenty steps. The work you submit should reflect the Maple Example above and should display the spreadsheet and display the approximation on the same plot with the direction field. Do not be concerned about the precise solution.

C2M9

Growth and Decay

The exponential function appears in many problems that involve physical phenomena. Newton's **Law of Cooling**, growth and decay, and the mixing of solutions come to mind as examples. In each case, the rate of change of some quantity is proportional to some aspect of the amount present. Imagine placing one steel bar at $32°$ Fahrenheit in a room that is $40°$ and another in a room that is $100°$. Certainly the temperature of the second bar will change more rapidly than that of the first. Newton's **Law of Cooling** asserts that the rate of change of the surface temperature of an object is proportional to the difference between that temperature and that of the surrounding medium. So if F is the temperature of the steel bar and R_T is the room temperature, then

$$\frac{dF}{dt} = k\,(F - R_T)$$

Example: Suppose that a steel bar is cooled to $-10°$ F and it is placed in a room that is $70°$ F. Ten minutes later the temperature of the bar is $25°$. Find an expression for the temperature at any time t and specifically at $t = 30$. By column

$\frac{dF}{dt} = k(F-70)$
$\frac{dF}{F-70} = k\,dt$
$\int \frac{dF}{F-70} = \int k\,dt$
$\ln|F-70| = kt + C$ but $|F-70| = 70 - F$
$70 - F = e^{kt+C}$ $F(0) = -10 \Rightarrow C = \ln 80$
$F = 70 - e^{kt+\ln 80} = 70 - 80e^{kt}$

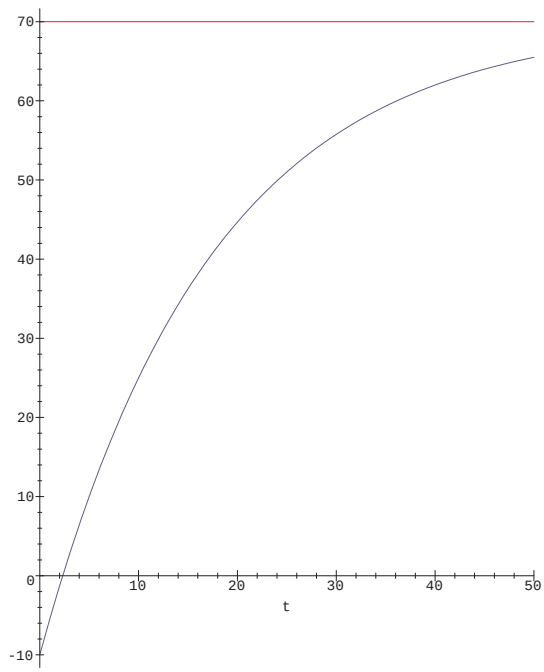
$F(10) = 25 = 70 - 80\,e^{10k}$
$80\,e^{10k} = 70 - 25 = 45$
$e^{10k} = \frac{45}{80} = \frac{9}{16}$
$10k = \ln\frac{9}{16}$
$k = -.0575364$
$F = 70 - 80e^{-.0575364\,t}$

which yields $F(30) = 55.7617°$. The exponential term goes to zero as time increases, so let's use Maple to graph F and see how F tends to 70.

```
> with(plots):
> F:=70-80*exp(-.0575364*t);
```

$$F := 70 - 80\,e^{(-.0575364\,t)}$$

```
> A:=plot(F,t=0..50):
> B:=plot(70,t=0..50):
> display(A,B);
```



Consider how the slope of the tangent line will change as a point moves from left to right along the curve. This means that the rate of change is a decreasing function, which is consistent with Newton's **Law of Cooling**.

C2M9 Problems: In problems 1 and 2, use Maple to plot the graph(s) of the given function(s) on the same axes. Remember that the exponential function e^x in Maple is `exp(x)`.

1. $f(t) = 60 + 30\,e^{-.3\,t}$ $g(t) = 60,$ $0 \le t \le 40$
2. $h(t) = 3\,e^{-t},\ j(t) = -4\,e^{-3\,t},\ h(t) + j(t),\ 0 \le t \le 6$
3. (Pencil and paper) A steel bar is heated to $200°$ and placed in a room whose temperature is $60°$. Thirty minutes later the bar is $110°$.

   (a) Find an expression for the temperature of the bar at any time t.

   (b) At what time will the temperature of the bar be $80°$?

C2M10

Sequences

When your instructor told you that a sequence is a function whose domain is the natural numbers, or a subset thereof, it is possible that you did not attach as much importance to that idea as you did to the mechanics of dealing with sequences. Maple allows us to define expressions and functions, and it is sometimes confusing as to which we want to use. We will define the sequence $\{a_n\} = \left\{\dfrac{n+2}{3n-1}\right\}$ as an expression and the sequence $\{b_n\} = \left\{\sqrt{n^2+3n} - n\right\}$ as a function to illustrate how they must be handled differently.

Maple Example 1: $\{a_n\} = \left\{\dfrac{n+2}{3n-1}\right\}$

```
> with(student):
> a:=(n+2)/(3*n-1);
```

$$a := \frac{n+2}{3n-1}$$

```
> seq(a,n=1..10);
```

$$\frac{3}{2}, \frac{4}{5}, \frac{5,}{8}, \frac{6}{11}, \frac{1}{2}, \frac{8}{17}, \frac{9}{20}, \frac{10}{23}, \frac{11,}{26}, \frac{12}{29}$$

```
> limit(a,n=infinity);
```

$$\frac{1}{3}$$

Maple Example 2: $\{b_n\} = \left\{\sqrt{n^2+3n} - n\right\}$

```
> with(student):
> b:=n->sqrt(n^2+3n)-n;
```

$$b := n \to \sqrt{n^2+3n} - n$$

```
> seq(b(n),n=1..10);
```

$$1, \sqrt{10}-2, 3\sqrt{2}-3, 2\sqrt{7}-4, 2\sqrt{10}-5, 3\sqrt{6}-6, \sqrt{70}-7, s\sqrt{22}-8, 6\sqrt{3}-9, \sqrt{130}-10$$

```
> limit(b(n),n=infinity);
```

$$\frac{3}{2}$$

To understand this last limit, consider multiplying b_n by its conjugate, and then dividing by it.

$$\left(\sqrt{n^2+3n} - n\right) \cdot \frac{\sqrt{n^2+3n}+n}{\sqrt{n^2+3n}+n} = \frac{n^2+3n-n^2}{\sqrt{n^2+3n}+n} = \frac{3n}{\sqrt{n^2+3n}+n} \cdot \frac{1/n}{1/n} = \frac{3}{\sqrt{1+3/n}+1} \to \frac{3}{2}$$

with the limit taken as $n \to \infty$.

The sequence $\{a_n\}$ is obtained from an expression whose name is a while the sequence $\{b_n\}$ is obtained by evaluating a function whose name is b. If we had the command `seq(b,n=1..10);` what would we have obtained? The answer - ten b's, because the function b must be evaluated in order for it to have a value. This will be very important in the next section when we will need to consider the term $\dfrac{b_{n+1}}{b_n} = \dfrac{b(n+1)}{b(n)}$. It would be cumbersome and less clear to find $\dfrac{a_{n+1}}{a_n}$ when a is an expression. The command would be `subs(n=n+1,a)/a;`.

C2M10 Problems Using Maple, find the first ten terms of each sequence and the limit of each.

1. $a_n = \left\{\dfrac{n^2}{3^n}\right\}$
2. $b_n = \left\{\dfrac{n^2-3n+4}{5+2n+6n^2}\right\}$
3. $c_n = \left\{\left(1+\dfrac{2}{n}\right)^n\right\}$
4. $d_n = \left\{\left(1-\dfrac{2}{n}\right)^n\right\}$
5. $e_n = \left\{\sqrt{n^2+6n} - n\right\}$

C2M11

Ratio Test

The ratio test is one of the most important tools in the study of infinite series. Its validity is a consequence of what we know about geometric series. For Maple purposes we will define the sequence upon which the series is based as a function of n. So we will use $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a(n)$ which emphasizes that $a_n = a(n)$ is really a function of n.

Example: Discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{n\,5^n}$.

We use the ratio test and consider $\frac{a_{n+1}}{a_n} = \frac{2^{2n+3}}{(n+1)\,5^{n+1}} \cdot \frac{n\,5^n}{2^{2n+1}} = \frac{2^2\,n}{5\,(n+1)}$. Take the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^2\,n}{5\,(n+1)} = \frac{4}{5}$ and conclude that the given series converges because the limit is less than one. Now, let's do this same problem using Maple. Note how we define $a_n = a(n)$ **as a function**, but the ratio, r_n, **is an expression**.

Maple Example:

```
> with(student):
> a:=n->2^(2*n+1)/n*5^n);
```

nth term of series

$$a := n \to \frac{2^{(2n+1)}}{n\,5^n}$$

```
> rn:=a(n+1)/a(n);
```

ratio for series

$$rn := \frac{2^{(2n+3)}\,n\,5^n}{(n+1)\,5^{(n+1)}2^{(2n+1)}}$$

```
> rn:=simplify(rn);
```

$$rn := \frac{4}{5}\frac{n}{n+1}$$

```
> limit(rn,n=infinity);
```

$$\frac{4}{5}$$

C2M11 Problems Use Maple to assist with the ratio test for the given series. Remember to include a concluding remark about the ratio test results.

1. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{3^n\,n!}$
2. $\sum_{n=1}^{\infty} \frac{(3n)!}{2^{2n}7^n(n!)^3}$
3. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

C2M12

Ratio Test for Power Series

We extend our use of Maple to power series by employing the same approach as in the previous section, but realizing that there will be a variable, or parameter, x involved. We may no longer assume that all the terms are positive, so the absolute value **must** be used.

Example: Find the open interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{2^n\,x^n}{n\,3^{n+1}}$.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}}{(n+1)\,3^{n+2}} \cdot \frac{n\,3^{n+1}}{2^n\,x^n}\right| = \frac{2}{3}\left|\frac{n}{n+1}\right| \cdot |x| \to \frac{2}{3}|x|$ with the limit taken as n goes to infinity and x is held constant. When are we guaranteed that this series will converge? When the limit of the ratio test is forced to be less than one. Thus, $\frac{2}{3}|x| < 1 \;\Rightarrow\; |x| < \frac{3}{2}$. We conclude that the series converges for all x in the open interval $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

Maple Example: Use Maple to find the open interval of convergence of the previous example.

```
> an:=n->(2^n*x^n)/(n*3^(n+1));
```

$$an := n \to \frac{2^n\, x^n}{n\, 3^{(n+1)}}$$

```
> rn:=an(n+1)/an(n);
```

$$rn := \frac{2^{(n+1)}\, x^{(n+1)}\, n\, 3^{(n+1)}}{(n+1)\, 3^{(n+2)} 2^n\, x^n}$$

```
> rn:=abs(simplify(rn));
```

$$rn := \frac{2}{3}\left|\frac{xn}{n+1}\right|$$

```
> limit(rn,n=infinity);
```

$$\frac{2}{3}|x|$$

```
> solve(%<1,x);
```

$$\mathrm{RealRange}\left(\mathrm{Open}\left(\frac{-3}{2}\right), \mathrm{Open}\left(\frac{3}{2}\right)\right)$$

So the open interval $(-3/2, 3/2)$ is our answer.

C2M12 Problems: Use Maple to find the open interval of convergence of the given power series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{(n+1)!}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(2n)! x^n}{(3n)!}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(2/3)^n (x+2)^n}{n^2}$

C2M13

Maclaurin and Taylor Series

It is remarkable that knowing about the values of a function and its derivatives at a point provides a means of evaluating the function at points nearby. Maclaurin and Taylor series are that means. Taylor and Maclaurin series are written respectively as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n\,!}(x-c)^n \qquad\qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n\,!} x^n$$

where, by letting $c = 0$, we see that the Maclaurin series is a special case of the Taylor series. The reader is reminded that

$$\begin{aligned}
0! &= 1 \\
1! &= 1 \cdot 0! = 1 \\
2! &= 2 \cdot 1! = 2 \\
3! &= 3 \cdot 2! = 6 \\
4! &= 4 \cdot 3! = 24 \\
5! &= 5 \cdot 4! = 120 \\
6! &= 6 \cdot 5! = 720 \\
7! &= 7 \cdot 6! = 5040 \\
8! &= 8 \cdot 7! = 40320 \\
9! &= 9 \cdot 8! = 362880 \\
10! &= 10 \cdot 9! = 3628800
\end{aligned}$$

Just for fun, in a Maple worksheet enter 357!;. The speed with which this computation is done is remarkable.

Frequently it is useful to write out the first few terms of a Taylor series. The result is a *Taylor Polynomial*. For example,

$$T_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maple syntax for a Taylor polynomial of degree n at $x = a$ is:

```
> taylor(f(x),x=a,n);
```

Every student should grind out a few Taylor series by hand so that they appreciate how the coefficients are determined. But, we can use Maple spreadsheets to accomplish the same thing. A worksheet entitled **Taylor Series Worksheet** follows that does exactly that. We will explain a few subtleties of the 'how's and why's' of this worksheet now.

Begin by defining the function $f(x) = \cos(2x)$ and identifying $a = \pi/6$. Recall that to open the spreadsheet you must click on **Insert** and the **Spreadsheet**. Resize it so that it shows about 10 rows and 4 columns. Enter n in cell **A1**, *function* in cell **B1** and $x = a$ in cell **C1**. Put 0 in **A2**, $f(x)$ in **B2**, and then `eval(~B2,x=a)` in **C2**. Continue by highlighting from **A2** down to **A8**, click on the button on the third row of the menus at the extreme left, insert a step size of 1, and then click **OK**. Move to cell **B3**. Enter `diff(~B2,x)` and then highlight that cell down to **B8**. Click on **Spreadsheet**, **Fill**, and **Down** and the successive derivatives should appear. Highlight from **C2** down to **C8**, and click on **Spreadsheet, Fill, Down** as before. Now the spreadsheet should be complete.

Continue by highlighting cells **C2** down to **C8**, clicking on the **Copy** button, clicking on the command line just below the spreadsheet, and then clicking on the **Paste** button. Important: put a colon at the end of MATRIX material and <Enter>. Convert the matrix to a list as shown, and then note that we have a list of small matrices. We want to formulate our coefficients for the Taylor series without the matrix brackets and this is done in our definition of the function b. Check out the numerator of the fraction. We have `c[n+1]` as the nth matrix in c if we start with 0. That matrix has one entry, and to access that number we use `(c[n+1])[1]`.

The coefficients for the Taylor polynomial, T_6, are $b(0), b(1), b(2), b(3), b(4), b(5), b(6)$, which we enter as $b(n)$ in the summation. We used $x - a$ which is easier than $x - Pi/6$. Then, let Maple do the same thing in one line. Compare the coefficients.

Taylor Series Worksheet

```
> restart:    with(student):

> f:=x->cos(2*x);
```

$$f := x \to \cos(2x)$$

```
> a:=Pi/6:
```

	A	B	C
1	n	*function*	$x = \frac{1}{6}\pi$
2	0	$\cos(2x)$	$\frac{1}{2}$
3	1	$-2\sin(2x)$	$-\sqrt{3}$
4	2	$-4\cos(2x)$	-2
5	3	$8\sin(2x)$	$4\sqrt{3}$
6	4	$16\cos(2x)$	8
7	5	$-32\sin(2x)$	$-16\sqrt{3}$
8	6	$-64\cos(2x)$	-32

```
> MATRIX([[1/2], [-sqrt(3)], [-2], [4*sqrt(3)], [8],[-16*sqrt(3)], [-32]]):

> c:=convert(%,listlist);
```

$$c := [[1/2], [-\sqrt{3}], [-2], [4\sqrt{3}], [8], [-16\sqrt{3}], [-32]]$$

```
> b:=n->(c[n+1])[1]/n!;
```

$$b := n- > \frac{c_{n+1_1}}{n!}$$

```
> T6:=sum(b(n)*(x-a)^n,n=0..6);
```

$$T6 := \frac{1}{2} - \sqrt{3}\left(x - \frac{1}{6}\pi\right) - \left(x - \frac{1}{6}\pi\right)^2 + \frac{2}{3}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^3 + \frac{1}{3}\left(x - \frac{1}{6}\pi\right)^4 - \frac{2}{15}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^5 - \frac{2}{45}\left(x - \frac{1}{6}\pi\right)^6$$

```
> taylor(f(x),x=a,7);
```

$$\frac{1}{2} - \sqrt{3}\left(x - \frac{1}{6}\pi\right) - \left(x - \frac{1}{6}\pi\right)^2 + \frac{2}{3}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^3 + \frac{1}{3}\left(x - \frac{1}{6}\pi\right)^4 - \frac{2}{15}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^5 - \frac{2}{45}\left(x - \frac{1}{6}\pi\right)^6 + O\left((x - \frac{1}{6}\pi\right)^7$$

C2M13 Problem: Use Maple and the method illustrated above to find a Taylor polynomial, T_6, for $f(x) = \arctan(x)$ at $a = 1$. Your work should display the spreadsheet, T_6, and the Maple solution.

C2M14

Animation of Taylor Polynomials

From our text we have seen that a function $f(x)$ may be written as the sum of an nth degree polynomial $T_n(x)$ and a remainder term $R_n(x)$ which tends to 0 as n approaches infinity. For an open interval $|x-a| < R$

$$f(x) = T_n(x) + R_n(x) \qquad T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \qquad \lim_{n\to\infty} R_n(x) = 0$$

It should be obvious to the reader at this point that T_n approximates f on the interval $|x - a| < R$ and that as n increases and we take more terms then the approximation improves. It is our objective here to provide a graphical illustration of this process. The function that we will use as our introductory example is $f(x) = \dfrac{1}{1+x}$. Our experience with geometric series allows us to "play Jeopardy" and pose "$\frac{1}{1+x}$ is the sum of what series?" Our response is "What is the sum of $1 - x + x^2 - x^3 + x^4 - x^5 \ldots$ " This allows us to easily identify $T_3(x) = 1 - x + x^2 - x^3$ as a simple approximation to $f(x)$. Now we will set all this up in Maple and display the functions.

```
> restart:      with(student):
> f:=x->1/(1+x);
```

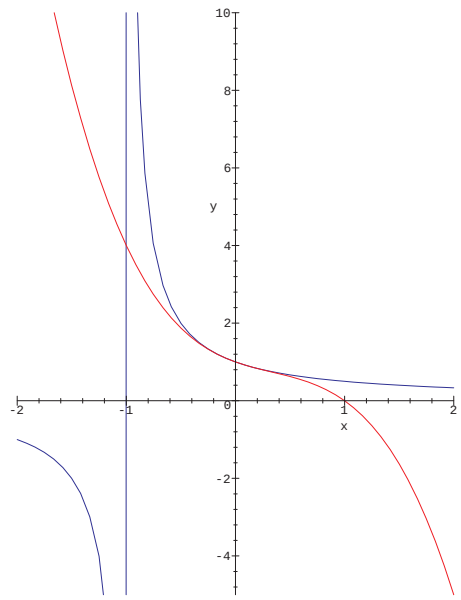
$$f := x \to \frac{1}{1+x}$$

We are going to set up values to serve as domain and range of our plots and an expression which is really $T_3(x)$.

```
> a:=-2:    b:=2:    c:=-5:    d:=10:
> P:=1-x+x^2-x^3;
```

$$P := 1 - x + x^2 - x^3$$

```
> plot([f(x),P], x=a..b,y=c..d,color=[blue,red],thickness=[1,2]);
```



Now we begin the process of setting up our sequence of Taylor Polynomials, which we do by identifying the degrees that we want to see displayed. We will select the first 29 Taylor Polynomials. So, we will start with 1 and increase by 1 until we reach 29. Continuing our worksheet,

```
> nstart:=1:    skip:=1:    frameno:=29:
> framenumbers:=[seq(nstart + skip*i, i=0..frameno-1)]:
```

$$framenumbers := [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]$$

Now we will set up our *frameno* (29) Taylor polynimial plots and they will appear in red.

```
> A:=display(seq(plot(convert(taylor(f(x),x=0,i),polynom,
    x=a..b,y=c..d,style=line,thickness=2,numpoints=100,
    title="degree = ".i),i=framenumbers),insequence=true):
```

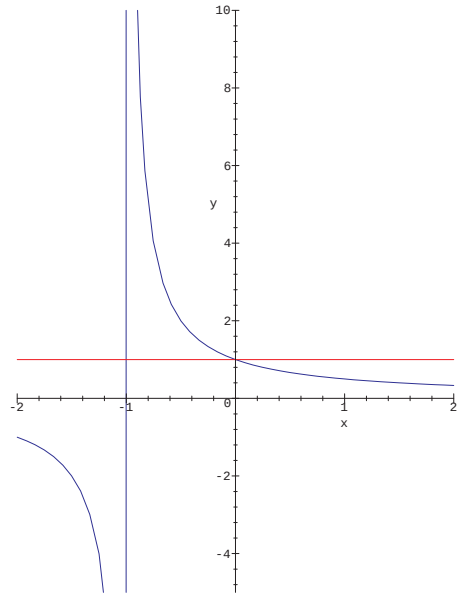
We need matching plots of $f(x)$, which will appear in blue.

```
> B:=animate(f(x),x=a..b,y=c..d,frames=frameno,color=blue):
```

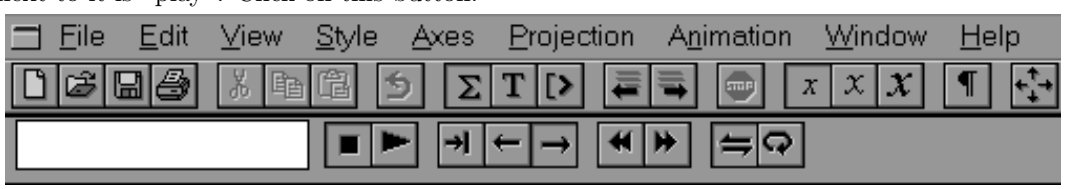
Combine all this into one plot.

```
> display(A,B,view=[a..b,c..d]);
```

degree = 1



You should see the first Taylor Polynomial, which is $y = 1$, displayed with the graph of $f(x)$. Click on the display and a box should appear around the display. Also, a new menu bar appears so that you have what looks like the buttons for a tape player on the bottom line. The button on the left is "stop" and the button next to it is "play". Click on this button.



You should see the rapid display of the Taylor Polynomials with $f(x)$. Now try the next button to the right, and click on it repeatedly.

If you just want to display a few frames of the animation, it is simpler to just identify the numbers of the frames. The lines below could replace the commands above to get a list of frame numbers.

```
> framenumbers:= [1,3,5,7,9,11,13,15,17,19,21,23,25,27,29];
```

$$framenumbers := [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29]$$

```
> frameno:=nops(framenumbers);
```

$$frameno := 15$$

The Maple command `nops` returns the number of elements in the list.

C3M14 Problems: 1. Use $-10 \le x \le 10$ and $-10 \le y \le 10$ to set up an animation for the Taylor Polynomials of $\sin(x)$ with $a = 0$.

2. Make up a Taylor Polynomial animation example using an interesting function of your choice.

C2M15

Vectors and the Scalar Product

The scalar, "dot", or "inner" product of two vectors is a very important concept that involves the size of each and the amount that each points in the direction of the other. This nebulous statement will have more meaning as you become familiar with vectors. In Maple, use `evalm` when evaluating a vector or matrix expression. Also, you will need the package *linalg*. In earlier versions of Maple one used `dotprod` to find the scalar product of two vectors. However, in Release 5 and 5.1 this command seems to be of better use when working with complex scalars. We are using only the real numbers, $\mathbb{R}$, so it is best to use the Maple command `innerprod` when finding the scalar product.

The concept of finding the projection of one vector on another is very important. First, the projection is a vector. Second, it is a scalar multiple of the vector being projected on, which must be a non-zero vector. When that scalar multiple is subtracted from the first vector the result is a vector that is orthogonal to the vector being projected on. Let's find the projection of $\overrightarrow{v_1} = \langle 2, 1, -3\rangle$ on $\overrightarrow{v_2} = \langle 1, -2, 2\rangle$. We will find the scalar c symbolically and then let Maple do the calculations and verify that it works. We seek c so that $(\overrightarrow{v_1} - c\overrightarrow{v_2}) \cdot \overrightarrow{v_2} = 0$. The *projection of* $\overrightarrow{v_1}$ *on* $\overrightarrow{v_2}$ is the vector $c\overrightarrow{v_2}$. Recall that two vectors are orthogonal (or perpendicular) if their scalar or dot product is 0.

$$\begin{aligned}(\overrightarrow{v_1} - c\overrightarrow{v_2}) \cdot \overrightarrow{v_2} &= \overrightarrow{v_1} \cdot \overrightarrow{v_2} - c\overrightarrow{v_2} \cdot \overrightarrow{v_2} = 0\\ \Longrightarrow \overrightarrow{v_1} \cdot \overrightarrow{v_2} &= c\overrightarrow{v_2} \cdot \overrightarrow{v_2}\\ \Longrightarrow c &= \frac{\overrightarrow{v_1} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}}\end{aligned}$$

The denominator $\overrightarrow{v_2} \cdot \overrightarrow{v_2}$ cannot be 0 because $\overrightarrow{v_2}$ is assummed to be non-zero. In Maple we have

```
> with(linalg):
> v1:=vector([2,1,-3]):  v2:=vector([1,-2,2]):
> c:=innerprod(v1,v2)/innerprod(v2,v2);
```

$$c := \frac{-2}{3}$$

```
> proj:=evalm(c*v2);
```

$$proj := \left[\frac{-2}{3}, \frac{4}{3}, \frac{-4}{3}\right]$$

```
> ortho:=evalm(v1-proj);
```

$$ortho := \left[\frac{8}{3}, \frac{-1}{3}, \frac{-5}{3}\right]$$

```
> innerprod(ortho,v2);
```

$$0$$

The "norm" or means of measuring the length of a vector that is most important to us is a special case of the so-called p-norm. We demonstrate for the vector $\overrightarrow{v} = \langle 2, -3\rangle$.

$$||\overrightarrow{v}||_p = (|2|^p + |-3|^p)^{1/p}$$

This has meaning if $p = 2$ and we will always use that value. So we will suppress the subscripts. This explanation is provided so that the '2' in the Maple command makes some sense. If you divide a non-zero vector by its length you have a vector of length one, called a "unit" vector. This process is called "normalizing" a vector. The vector $\overrightarrow{u}$ below is a unit vector in the direction of $\overrightarrow{v}$.

$$||\overrightarrow{v}||_2 = ||\overrightarrow{v}|| = ||\langle 2, -3\rangle|| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13} \qquad \overrightarrow{u} = \frac{1}{\sqrt{13}}\langle 2, -3\rangle$$

Returning to the Maple worksheet that we started above, let's compute $||\overrightarrow{v_2}||$ and normalize $\overrightarrow{v_2}$. The Maple command `normalize` assumes that the '2' norm is being used.

```
> lengthv2:=norm(v2,2);
```

$$lengthv2 := 3$$

```
> u2:=normalize(v2);
```

$$u2 := \left[\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right]$$

Recall that $\overrightarrow{a} \cdot \overrightarrow{b} = ||\overrightarrow{a}||\,||\overrightarrow{b}||\cos\theta$, where θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$. Obviously, for non-zero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ we can solve for $\cos\theta$ and thereby determine θ.

$$\cos\theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}||\,||\overrightarrow{b}||}$$

Maple saves us the work with the command "angle".

```
> theta:=angle(v1,v2);
```

$$\theta := \pi - \arccos\left(\frac{1}{21}\sqrt{14}\sqrt{9}\right)$$

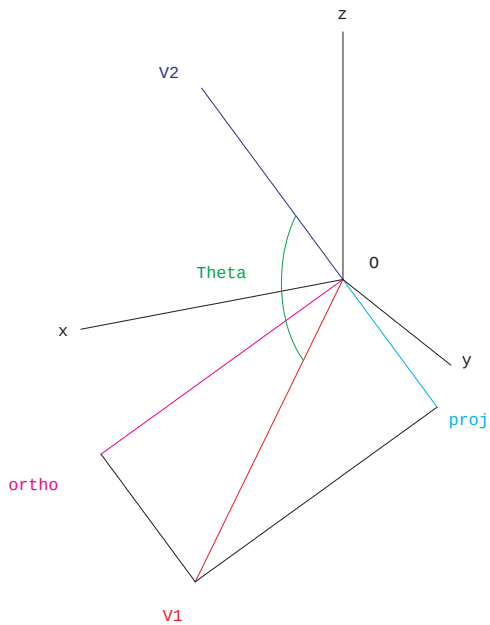
```
> theta:=evalf(theta);
```

$$\theta := 2.134738969$$

```
> theta1:=evalf(theta*180/Pi);
```

$$\theta 1 := 122.3115332$$

This makes sense in the following diagram where we see that θ must be an obtuse angle since the projection is opposite to the direction of $\overrightarrow{v_2}$.



The projection of $\overrightarrow{v1}$ on $\overrightarrow{v2}$

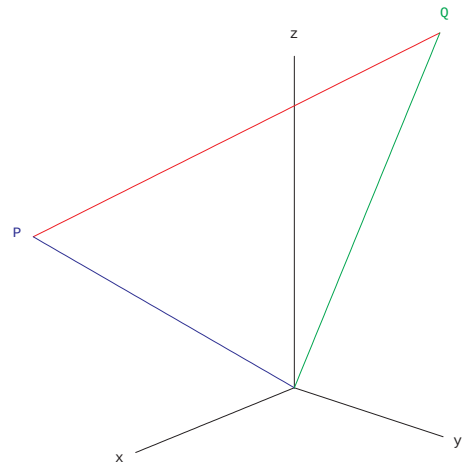
A line between two points P and Q is parameterized by $\alpha(t) = (1-t)\overrightarrow{P} + t\overrightarrow{Q}$ for $0 \le t \le 1$. We will use this in combination with `spacecurve` to plot lines.

Example: Plot the coordinate axes and the line segment from $P(4,-1,2)$ to $Q(1,4,4)$.

```
> with(plots):
> P:=vector([4,-1,2]):  Q:=vector([1,4,4]):
> xaxis:=spacecurve([t,0,0],t=0..3,color=black):
> yaxis:=spacecurve([0,t,0],t=0..3,color=black):
> zaxis:=spacecurve([0,0,t],t=0..3,color=black):
> alpha:=evalm((1-t)*P+t*Q);
```

$$\alpha := [4-3t, -1+5t, 2+2t]$$

```
> LPQ:=spacecurve(alpha,t=0..1,color=red):
> LP:=spacecurve(evalm(t*P),t=0..1,color=blue):
> LQ:=spacecurve(evalm(t*Q),t=0..1,color=green):
> display(xaxis,yaxis,zaxis,LPQ,LP,LQ);
```



C2M15 Problems:

1. Use Maple to find the projection of $\overrightarrow{v} = \langle 3, -3, 5\rangle$ on $\overrightarrow{w} = \langle -6, 3, 2\rangle$. Find a unit vector in the direction of $2\overrightarrow{v} - \overrightarrow{w}$.

2. Use Maple to find the angle between $\overrightarrow{v} = \langle 2, 2, -1\rangle$ and $\overrightarrow{w} = \langle 3, -4, 5\rangle$. Find a unit vector in the direction of $-\overrightarrow{v}$.

3. Use Maple to plot the line segment from $P(4, 1, 4)$ to $Q(-1, 3, 1)$ and the segments from P and Q to the origin. Include the coordinate axes as shown above. Use colors.

C2M16

Lines, Planes, and Distances in $\mathbb{R}^3$

This section deals with problems finding equations of lines and planes and distances from points to these objects. We will begin with a couple of simple examples.

Example 1: Suppose that we are given three points: $P(2, 1, -1)$, $Q(-1, 1, 0)$, and $Q(1, 3, -1)$, and we wish to find an equation for the plane that contains them and the area of the triangle that they form.

Begin by subtracting one point from the other two and finding the cross product of the results. That cross product will serve as a normal vector to our plane. Every vector in the plane will have the same scalar product with that normal vector. If $\overrightarrow{N}$ is the normal vector, $\overrightarrow{X_0}$ is a specific vector in the plane, $\overrightarrow{X}$ is any vector in the plane, then $\overrightarrow{X} - \overrightarrow{X_0}$ must be orthogonal (perpendicular) to $\overrightarrow{N}$. So the basic equation for the plane becomes

$$\overrightarrow{N} \cdot (\overrightarrow{X} - \overrightarrow{X_0}) = 0 \qquad \text{or equivalently} \qquad \overrightarrow{N} \cdot \overrightarrow{X} = \overrightarrow{N} \cdot \overrightarrow{X_0}$$

And in Maple,

```
> restart:    with(student):    with(linalg):
> P:=vector([2,1,-1]):  Q:=vector([-1,1,0]):  R:=vector([1,3,-1]):
> QP:=evalm(P-Q);
```

$$QP := [3, 0, -1]$$

```
> QR:=evalm(R-Q);
```

$$QR := [2, 2, -1]$$

```
> N:=crossprod(QP,QR);
```

$$N := [2, 1, 6]$$

```
> X:=vector([x,y,z]);
```

$$X := [x, y, z]$$

```
> plane:=innerprod(N,X)=innerprod(N,P);
```

$$plane := 2x + y + 6z = -1$$

```
> innerprod(N,Q);
```

$$-1$$

```
> innerprod(N,R);
```

$$-1$$

We used $\overrightarrow{P}$ as the known vector in the plane. After we found the equation we took the scalar product of $\overrightarrow{N}$ with $\overrightarrow{Q}$ and $\overrightarrow{R}$ just to check and see if we got the same value. With Maple doing the calculations for us no error occurred, but when doing this very basic problem with pencil and paper it is important to check your answer.

Now let's find the area of $\triangle PQR$. The two vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ generate a parallelogram whose area is $\|\overrightarrow{QP} \times \overrightarrow{QR}\| = \|\overrightarrow{N}\|$ and the triangle is one-half of that parallelogram.

```
> area:=norm(N,2)/2;;
```

$$area := \frac{1}{2}\sqrt{41}$$

A line in $\mathbb{R}^3$ requires a direction and a point on the line. If $\overrightarrow{N} = \langle a, b, c\rangle$ is the direction vector, $\overrightarrow{X_0} = \langle x_0, y_0, z_0\rangle$ is a specific point on the line, t is a scalar (parameter) and $\overrightarrow{X} = \langle x, y, z\rangle$ is any point on the line, then the parametric equation of the line is

$$\overrightarrow{X} = \overrightarrow{X_0} + t\,\overrightarrow{N} \qquad \text{or equivalently} \qquad \begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

If we want a parametric equation of a line **normal** (perpendicular, orthogonal) to our plane above and passing through P then

```
> nline:=evalm(X=P+t*N);
```

$$nline := [x, y, z] = [2 + 2t, 1 + t, -1 + 6t]$$

To obtain a parametric equation for the line through P in the direction of Q, we simply use $\overrightarrow{QP}$ as the direction vector.

```
> line:=evalm(X=P+t*QP);
```

$$line := [x, y, z] = [2 + 3t, 1, -1 - t]$$

Let's turn our attention to the distance from a point to a plane. Suppose $ax + by + cz = d$ is an equation for the plane and P is a point. You know that the coefficients a, b, c are attitude numbers for the plane and that $\overrightarrow{N} = \langle a, b, c\rangle$ is a normal vector. Find any point X_0 which lies in the plane (X_0 satisfies the equation). Form the vector $\overrightarrow{\alpha} = \overrightarrow{P - X_0}$. The distance from the point P to the plane will be the length of the projection of $\overrightarrow{\alpha}$ on $\overrightarrow{N}$.

$$\text{distance} = \|c\overrightarrow{N}\| = \left\| \frac{\overrightarrow{\alpha} \cdot \overrightarrow{N}}{\overrightarrow{N} \cdot \overrightarrow{N}} \overrightarrow{N} \right\| = \frac{|\overrightarrow{\alpha} \cdot \overrightarrow{N}|}{\|\overrightarrow{N}\|^2} \|\overrightarrow{N}\| = \frac{|\overrightarrow{\alpha} \cdot \overrightarrow{N}|}{\|\overrightarrow{N}\|}$$

Please note the diagram for this later in the section.

Example 2: Find the distance from $P = (4, -3, 5)$ to the plane with equation $2x + y + 6z = -1$.

We observe that $X_0 = (-1, 1, 0)$ lies in the plane and that $\overrightarrow{N} = \langle 2, 1, 6\rangle$ is a normal vector to this plane.

```
> N:=vector([2,1,6]):  P:=vector([4,-3,5]):  X0:=vector([-1,1,0]):
> alpha:=evalm(P-X0);
```

$$\alpha := [5, -4, 5]$$

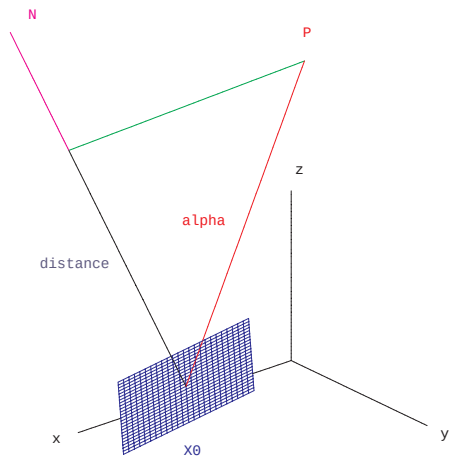
```
> distance:=abs(innerprod(alpha,N))/norm(N,2);
```

$$distance := \frac{36}{41}\sqrt{41}$$

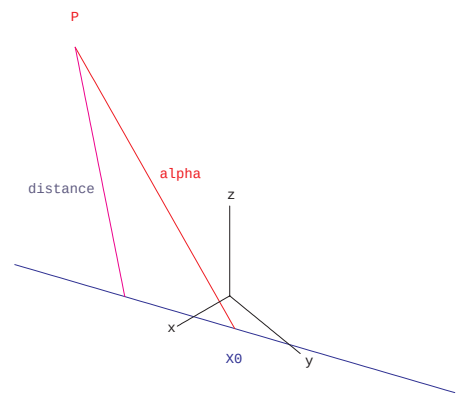
Let's turn our attention to finding the distance from a point to a line. If P is the point and $\overrightarrow{X} = \overrightarrow{X_0} + t\,\overrightarrow{N}$ is a parametric equation for the line, then it is easy to find X_0 which is a specific point on the line. Let $\overrightarrow{\alpha} = P - X_0$. An examination of the diagram and a little trigonometry shows that the distance $d = \|\overrightarrow{\alpha}\| \sin\theta$ where θ is the angle between $\overrightarrow{\alpha}$ and $\overrightarrow{N}$. Recall that for vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, $\|\overrightarrow{A} \times \overrightarrow{B}\| = \|\overrightarrow{A}\|\,\|\overrightarrow{B}\| \sin\theta$. This leads to

$$d = \|\overrightarrow{\alpha}\| \sin\theta = \|\overrightarrow{\alpha}\| \sin\theta \frac{\|\overrightarrow{N}\|}{\|\overrightarrow{N}\|} = \frac{\|\overrightarrow{\alpha}\|\,\|\overrightarrow{N}\| \sin\theta}{\|\overrightarrow{N}\|} = \frac{\|\overrightarrow{\alpha} \times \overrightarrow{N}\|}{\|\overrightarrow{N}\|}$$

Please note the similarities and contrasts between our two distance problems. Diagrams for each follow.



distance from point P to plane



distance from point P to line

Example 3: Find the distance from the point $P = (4, -3, 5)$ to the line with parametric equations

$$\begin{aligned} x &= -1 + 3t \\ y &= 1 + 5t, \\ z &= \quad + 4t \end{aligned}$$

We identify $\overrightarrow{N} = \langle -3, 5, 4 \rangle$ and $X_0 = (-1, 1, 0)$. In Maple

```
> N:=vector([-3,5,4]):  X0:=vector([-1,1,0]):  P:=vector([4,-3,5]):
> alpha:=evalm(P-X0);
```

$$\alpha := [5, -4, 5]$$

```
> CP:=crossprod(alpha,N);
```

$$CP := [-41, -35, 13]$$

```
> d:=norm(CP,2)/norm(N,2);
```

$$d := \frac{1}{2}\sqrt{123}\sqrt{2}$$

Exercises (pencil and paper):

1. Find an equation for the plane that contains the points $P(2, 3, -1)$, $Q(-1, 1, 4)$, $R(0, 1, 3)$ and the area of the triangle that they form. Find parametric equations for the line through P and Q.

2. Find an equation for the plane that contains the points $P(1, -1, 1)$, $Q(0, 2, -1)$, $R(3, 3, -1)$ and the area of the triangle that they form. Find parametric equations for the line through P and Q.

3. Find the distance from the point $P(3, 3, -2)$ to the plane whose equation is $2x - 3y + 2z = 7$.

4. Find the distance from the point $Q(3, 2, -1)$ to the plane whose equation is $x + 4y - z = 6$.

5. Find the distance from the point $P(3, 3, -2)$ to the line through $Q(1, 2, -1)$ and $R(3, -1, 5)$ and find parametric equations for the line.

6. Find the distance from the point $P(-2, 1, 3)$ to the line through $Q(0, 2, -1)$ and $R(-1, 4, 1)$ and find parametric equations for the line.

C2M16 Problems Use Maple to solve the problems 1, 3, and 5 above.

SUMMARY OF MAPLE COMMANDS

Using the package 'student'

Maple Command	Output
`area:=Pi*r^2;`	$area := \pi r^2$ Sets πr^2 as an expression in r named $area$
`f:=x->sqrt(4+9*x^2);`	$f := x \to \sqrt{4+9x^2}$ Sets $\sqrt{4+9x^2}$ as a function named $f(x)$
`subs(r=5,area);`	25π Substitutes 5 into the expression $area$ as r
`f(1);`	$\sqrt{13}$ Evaluates the function f at 1
`eval(expr,x=a);`	Evaluates the expression *expr* at $x = a$
`plot(area,r=1..4);`	Produces a plot of expression $area$, r ranges from 1 to 4
`plot(f(x),x=-2..5);`	Produces a plot of function $f(x)$, x ranges from -2 to 5
`diff(area,r);`	$2\pi r$ The derivative of the expression $area$ with respect to r
`diff(f(x),x);`	$9\frac{x}{\sqrt{4+9x^2}}$ The derivative of the function $f(x)$ with respect to x
`int(area,r);`	$\frac{1}{3}\pi r^3$ The anti-derivative of the expression $area$ with respect to r
`int(area,r=1..4);`	21π The definite integral of $area$ from 1 to 4.
`evalf(%);`	65.97344573 The decimal form of the previous expression, 21π
`Int(area,r);`	$\int \pi\, r^2\, dr$ The inert expression whose value is the antiderivative of $area$
`sum(f(k),k=n1..n2);`	Evaluates the sum $\sum_{k=n1}^{k=n2} f(k)$ of $f(k)$ from $n1$ to $n2$
`leftbox(f(x),x=a..b,n);`	Displays the Riemann sum graphically for $f(x)$ over the interval $[a,b]$ using n intervals and left endpoints
`rightsum(f(x),x=a..b,n);`	Displays the Riemann sum in summation form for $f(x)$ over the interval $[a,b]$ using n intervals and right endpoints

Using the package 'plots'

`spacecurve([f(t),g(t),h(t)],t=a..b);`	Displays the three-dimensional curve defined parametrically by three expressions in t, which ranges from a to b
`plot3d(expr,var1=a..b,var2=c..d);`	Displays a surface for an expression in two variables
`plot3d([expr1,expr2,expr3],var1=a..b, var2=c..d);`	Displays a surface parametrically as a function of two variables
`cylinderplot(expr,var1=a..b,var2=c..d);`	Displays $z =$expr in terms of $r =$var1 and $\theta =$var2
`polarplot(expr,var=a..b);`	Displays a polar coordinate plot of $r =$expr in terms of $\theta =$var
`sphereplot(expr,var1=a..b,var2=c..d);`	Displays $\rho =$expr in terms of $\theta =$var1 and $\phi =$var2
`cylinderplot([expr1,expr2,expr3], var1=a..b,var2=c..d);`	Displays a 3-d cylindrical plot parametrically, (r, θ, z)
`polarplot([expr1,expr2,var=a..b]);`	Displays a parametric polar coordinate plot of $r =$expr1 and $\theta =$expr2
`sphereplot([expr1,expr2,expr3], var1=a..b,var2=c..d);`	Displays a 3-d spherical plot parametrically, (ρ, θ, ϕ)
`implicitplot(equation,var1=a..b, var2=c..d);`	Displays a 2-d implicit plot of equation
`implicitplot3d(eqn,var1=a..b, var2=c..d,var3=e..f);`	Displays a 3-d plot for the equation eqn
`display({Plot1,Plot2,..,Plotn});`	Displays n plots on the same coordinate system

Using the package 'linalg'

`A:=vector([a,b,c]);`	Assigns the name A to the vector $\langle a, b, c\rangle$
`A[1]; A[2]; A[3];`	The components of the vector A are displayed
`F:=vector([expr1,expr2,expr3]);`	Assigns the name F to the vector expression
`map(diff,F,t);`	Gives the derivative of the vector $\overrightarrow{F}$ with respect to t
`innerprod(A,B);`	Finds $\overrightarrow{A} \cdot \overrightarrow{B}$, the scalar product of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$
`crossprod(A,B);`	Finds $\overrightarrow{A} \times \overrightarrow{B}$, the cross product of vectors $\overrightarrow{A}$ and $\overrightarrow{B}$
`evalm(A+c*B);`	Finds $A + cB$ for vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, and scalar c
`hessian(f,[x,y]);`	Finds the hessian matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ for the function f of x and y
`det(A);`	Finds the determinant of the square matrix A
`subs({x=a,y=b},op(H));`	Substitutes values for x and y into components of matrix/vector H